Mendler-Style Inductive Types, Categorically

Extended Abstract

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Abstract We present a basis for a categorical account of Mendler-style inductive types by introducing a notion of initial Mendler-style algebras and use it for giving a reduction of (conventional) inductive types to Mendler-style inductive types and two reductions of Mendler-style inductive types to (conventional) inductive types.

1 Introduction

For a category-theoretically minded computing scientist, the phrase "inductive types" normally serves as a name for a construction that is usually conveniently modelled by means of the notion of initial algebras [Mal90,Fok92]. In typed lambda calculi (type theories), at the same time, one encounters both inductive types in this (call it conventional) sense and Mendler-style inductive types [Men91,Lei90,Uus98,Mat98]. This paper reports the first results of an ongoing work on developing a categorical account of Mendler-style inductive types, based on a notion of initial Mendler-style algebras invented for this purpose and motivated from an interest in generic programming.

The notion of inductive types is defined for covariant type constructors. The lattice-theoretic prototype of an inductive type is a least prefixed point of a monotone function f, i.e. a least point among the points c such that such that $fc \leq c$. The notion of Mendler-style inductive types is defined in this paper for multivariant (i.e., di-) type constructors, but it seems useful only for covariant type constructors still. The lattice-theoretic prototype of a Mendler-style inductive type is, for an arbitrary function f, a least point among the points c such that, for any point a, if $a \leq c$, then $fa \leq c$. If, for any point c, there exists a lub $f^{\mathrm{e}}c$ of the set $\{fa | a \leq c\}$, then f^{e} is a monotone function and this point is, in fact, a least prefixed point of f^{e}.

Throughout the text, we work with an anonymous base category, which is not required to have any specific structure. The paper is written in the spirit of the constructive algorithmics community. So, instead diagram chasing, our proof discipline is structured calculation.

2 Inductive types: a reminder

The standard categorical account of inductive types is based on the notion of initial algebras.

Let F be a covariant (endo)functor. Any pair (C, ϕ) formed of an object C and a morphism $\phi : FC \to C$ is said to be an *F-algebra*. For any two F-algebras (C, ϕ) and (D, ψ), any morphism $h : C \to D$ such that $h \circ \phi = \psi \circ Fh$ is said to be a *(F-algebra) homomorphism* from (C, ϕ) to (D, ψ).

A pair $((\mu F, \mathtt{In}_F), (\!| \cdot |\!)_F)$ is called an *initial F-algebra* (an inductive datatype), if $(\mu F, \mathtt{In}_F)$ is an F-algebra and, for any other F-algebra (C, ϕ), the entity $(\!| \phi |\!)_F$ is a unique homomorphism from $(\mu F, \mathtt{In}_F)$ to (C, ϕ), i.e., if the following judgements hold:

$(\mu F, \mathtt{In}_F)$**-prop**

$$\mu F : \mathrm{Obj},\ \mathtt{In}_F : F(\mu F) \to \mu F$$

$(\!| \cdot |\!)_F$**-char-exi**

$$\begin{array}{l} C : \mathrm{Obj},\ \phi : FC \to C \\ \ \Rightarrow\ (\!| \phi |\!)_F : \mu F \to C,\ (\!| \phi |\!)_F \circ \mathtt{In}_F = \phi \circ F(\!| \phi |\!)_F \end{array}$$

$(\!| \cdot |\!)_F$**-char-uniq**

$$\begin{array}{l} C : \mathrm{Obj},\ \phi : FC \to C \\ \ \Rightarrow\ (\ f : \mu F \to C,\ f \circ \mathtt{In}_F = \phi \circ Ff \\ \qquad \Rightarrow\ \ f = (\!| \phi |\!)_F\ ) \end{array}$$

3 Mendler-style inductive types: a news?

To be able to model Mendler-style inductive types categorically, we introduce the notion of initial Mendler-style algebras.

Let F be a di-(endo-)functor. Any pair (C, Φ) formed of an object C and a morphism-with-codomain-C mapping Φ is said here to be a *Mendler-style F-algebra*, if the following "extra" condition (below referred

to by the name (C,Φ)-F-extra) is met: (0) for any morphism $\alpha :\to C$, $\mathtt{dom}\,(\Phi\alpha) = F(\mathtt{dom}\,\alpha, \mathtt{dom}\,\alpha)$, and (1) for any morphisms $\alpha :\to C$, $\beta :\to C$, $g : \mathtt{dom}\,\alpha \to \mathtt{dom}\,\beta$, if $\alpha = \beta \circ g$, then $\Phi\alpha \circ F(g, \mathtt{id}_{\mathtt{dom}\,\alpha}) = \Phi\beta \circ F(\mathtt{id}_{\mathtt{dom}\,\beta}, g)$.

For any two Mendler-style F-algebras (C,Φ), (D,Ψ), any morphism $h : C \to D$ such that, for any morphism α with codomain C, $h \circ \Phi\alpha = \Psi(h \circ \alpha)$, is said to be a *(Mendler-style F-algebra) homomorphism* from (C,Φ) to (D,Ψ).

A pair $((\mu^{\mathrm{m}}F, \mathtt{In}^{\mathrm{m}}_F), (\!|\cdot|\!)^{\mathrm{m}}_F)$ is called an *initial Mendler-style F-algebra* (a Mendler-style inductive datatype), if $(\mu^{\mathrm{m}}F, \mathtt{In}^{\mathrm{m}}_F)$ is a Mendler-style F-algebra and, for any other Mendler-style F-algebra (C,Φ), the entity $(\!|\,\Phi\,|\!)^{\mathrm{m}}_F$ is a unique homomorphism from $(\mu^{\mathrm{m}}F, \mathtt{In}^{\mathrm{m}}_F)$ to (C,Φ), i.e., if the following judgements hold:

$(\mu^{\mathrm{m}}F, \mathtt{In}^{\mathrm{m}}_F)$-prop

$$\mu^{\mathrm{m}}F : \mathrm{Obj},\ (\, m :\to \mu^{\mathrm{m}}F \ \Rightarrow\ \mathtt{In}^{\mathrm{m}}_F m :\to \mu^{\mathrm{m}}F\,),\ (\mu^{\mathrm{m}}F, \mathtt{In}^{\mathrm{m}}_F)\text{-}F\text{-extra}$$

$(\!|\cdot|\!)^{\mathrm{m}}_F$-char-exi

$$\begin{array}{l} C : \mathrm{Obj},\ (\, \gamma :\to C \ \Rightarrow\ \Phi\gamma :\to C\,),\ (C,\Phi)\text{-}F\text{-extra} \\ \quad \Rightarrow\ (\!|\,\Phi\,|\!)^{\mathrm{m}}_F : \mu^{\mathrm{m}}F \to C,\ (\, m :\to \mu^{\mathrm{m}}F \ \Rightarrow\ (\!|\,\Phi\,|\!)^{\mathrm{m}}_F \circ \mathtt{In}^{\mathrm{m}}_F m = \Phi((\!|\,\Phi\,|\!)^{\mathrm{m}}_F \circ m)\,) \end{array}$$

$(\!|\cdot|\!)^{\mathrm{m}}_F$-char-uniq

$$\begin{array}{l} C : \mathrm{Obj},\ (\, \gamma :\to C \ \Rightarrow\ \Phi\gamma :\to C\,),\ (C,\Phi)\text{-}F\text{-extra} \\ \quad \Rightarrow\ (\, f : \mu^{\mathrm{m}}F \to C,\ (\, m :\to \mu^{\mathrm{m}}F \ \Rightarrow\ f \circ \mathtt{In}^{\mathrm{m}}_F m = \Phi(f \circ m)\,) \\ \qquad \Rightarrow\ f = (\!|\,\Phi\,|\!)^{\mathrm{m}}_F\,) \end{array}$$

4 Inductive types reduced to Mendler-style inductive types

Let F be a covariant functor. Define a binary object and morphism mapping $F^{\wr}$ (a padding of F with a dummy argument place) as follows:

$F^{\wr}$-def

$$\begin{array}{r} C' : \mathrm{Obj}, C : \mathrm{Obj} \ \Rightarrow\ F^{\wr}(C', C) = FC \\ h' :\to, h :\to \ \Rightarrow\ F^{\wr}(h', h) = Fh \end{array}$$

It is obvious that $F^{\wr}$ is a difunctor.

Let us assume that there exists an initial Mendler-style $F^{\wr}$-algebra $((\mu^{\mathrm{m}}F^{\wr}, \mathtt{In}^{\mathrm{m}}_{F^{\wr}}), (\!|\cdot|\!)^{\mathrm{m}}_{F^{\wr}})$. Define

$(\mu F, \mathtt{In}_F)$**-def**

$$\mu F = \mu^{\mathrm{m}} F^l,\ \mathtt{In}_F = \mathtt{In}^{\mathrm{m}}_{F^l} \mathtt{id}_{\mu^{\mathrm{m}} F^l}$$

$(\!|\cdot|\!)_F$**-def**

$$C : \mathrm{Obj}, \phi : FC \to C \ \Rightarrow\ (\!|\, \phi \,|\!)_F = (\!|\, (\gamma : \to C)\, \phi \circ F\gamma \,|\!)^{\mathrm{m}}_{F^l}$$

(For $(\!|\cdot|\!)_F$-def to be a legitimate definition, the condition $(C, (\gamma : \to C)\, \phi \circ F\gamma)$-$F^l$-extra has to be true, which, luckily, is the case). It appears that the pair $((\mu F, \mathtt{In}_F), (\!|\cdot|\!)_F)$ is an initial F-algebra.

Proof of $(\!|\cdot|\!)_F$-char-exi:

$$\begin{array}{|l}
\rhd\ C : \mathrm{Obj}, \phi : FC \to C \\ \hline
\quad (\!|\, \phi \,|\!)_F \circ \mathtt{In}_F \\
= \quad -\ (\!|\cdot|\!)_F\text{-def},\ (\mu F, \mathtt{In}_F)\text{-def}\ - \\
\quad (\!|\, (\gamma : \to C)\, \phi \circ F\gamma \,|\!)^{\mathrm{m}}_{F^l} \circ \mathtt{In}^{\mathrm{m}}_{F^l} \mathtt{id}_{\mu^{\mathrm{m}} F^l} \\
= \quad -\ (\!|\cdot|\!)^{\mathrm{m}}_{F^l}\text{-char-exi}\ - \\
\quad \phi \circ F (\!|\, (\gamma : \to C)\, \phi \circ F\gamma \,|\!)^{\mathrm{m}}_{F^l} \\
= \quad -\ (\!|\cdot|\!)_F\text{-def}\ - \\
\quad \phi \circ F (\!|\, \phi \,|\!)_F
\end{array}$$

Proof of $(\!|\cdot|\!)_F$-char-uniq:

$$\begin{array}{|l}
\rhd\ C : \mathrm{Obj}, \phi : FC \to C \\ \hline
\rhd\ f : \mu F \to C, f \circ \mathtt{In}_F = \phi \circ F f \\ \hline
\quad f \\
= \quad -\ (\!|\cdot|\!)^{\mathrm{m}}_{F^l}\text{-char-uniq}\ - \\
\qquad \begin{array}{|l}
\rhd\ m : \to \mu^{\mathrm{m}} F^l \\ \hline
\quad f \circ \mathtt{In}^{\mathrm{m}}_{F^l} m \\
= \quad -\ F^l \text{ cov functor}\ - \\
\quad f \circ \mathtt{In}^{\mathrm{m}}_{F^l} m \circ F \mathtt{id}_{\mathtt{dom}\, m} \\
= \quad -\ F^l\text{-def}\ - \\
\quad f \circ \mathtt{In}^{\mathrm{m}}_{F^l} m \circ F^l (m, \mathtt{id}_{\mathtt{dom}\, m}) \\
= \quad -\ (\mu^{\mathrm{m}} F^l, \mathtt{In}^{\mathrm{m}}_{F^l})\text{-prop}\ - \\
\quad f \circ \mathtt{In}^{\mathrm{m}}_{F^l} \mathtt{id}_{\mu^{\mathrm{m}} F^l} \circ F^l (\mathtt{id}_{\mu^{\mathrm{m}} F^l}, m) \\
= \quad -\ (\mu F, \mathtt{In}_F)\text{-def},\ F^l\text{-def}\ - \\
\quad f \circ \mathtt{In}_F \circ F m \\
= \quad -\ \lhd\ - \\
\quad \phi \circ F f \circ F m \\
= \quad -\ F \text{ cov functor}\ - \\
\quad \phi \circ F (f \circ m)
\end{array} \\
\quad (\!|\, (\gamma : \to C)\, \phi \circ F\gamma \,|\!)^{\mathrm{m}}_{F^l} \\
= \quad -\ (\!|\cdot|\!)_F\text{-def}\ - \\
\quad (\!|\, \phi \,|\!)_F
\end{array}$$

an elegant theory can be developed for it. It is also obvious that the whole development readily dualizes for Mendler-style coinductive types.

The "extra" conditions of this paper are not ad hoc, they are, in fact, dinaturality conditions for transformations in derived categories, hence their truth can often be taken for granted ("free theorems"); the more exact implications of works on dinaturality [BFSS90,Fre93,Tui95] for our work is a subject of further investigation.

A specifically nice feature of Mendler-style inductive types from the programming perspective is that the "map" function (or the "comap" and "map" pair of functions) corresponding to the type constructor is not mentioned manifestly in the β (cancellation), η (reflection), and commutative (fusion) rules; they should thus provide support for "truly generic" programming, i.e. writing generic programs without having to involve induction over type constructor expressions. Also this issue needs a closer study.

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Proof of $(\!|\cdot|\!)^{\mathrm{m}}_{F}$-char-uniq:

$$
\begin{array}{l}
\triangleright\, C : \mathrm{Obj},\ (\,\gamma :\to C \ \Rightarrow\ \Phi\gamma :\to C\,),\ (C,\Phi)\text{-}F\text{-extra} \\
\hline
\triangleright\, f : \mu^{\mathrm{m}}F \to C,\ (\,m :\to \mu^{\mathrm{m}}F \ \Rightarrow\ f \circ \mathtt{In}^{\mathrm{m}}_{F} m = \Phi(f \circ m)\,) \\
\hline
\quad f \\
= \quad -\ (\!|\cdot|\!)_{F^{\mathrm{e}}}\text{-char-uniq}\ - \\
\qquad f \circ \mathtt{In}_{F^{\mathrm{e}}} \\
\qquad = \quad -\ [\,\cdot\,]\text{-char-uniq}\ - \\
\qquad\quad \triangleright\, m :\to \mu F^{\mathrm{e}} \\
\qquad\quad \overline{\quad f \circ \mathtt{In}_{F^{\mathrm{e}}} \circ \mathtt{in}_{F^{\triangleright}(\mu F^{\mathrm{e}})} m} \\
\qquad\quad = \quad -\ (\mu^{\mathrm{m}}F, \mathtt{In}^{\mathrm{m}}_{F})\text{-def}\ - \\
\qquad\quad\quad f \circ \mathtt{In}^{\mathrm{m}}_{F} m \\
\qquad\quad = \quad -\ \lhd\ - \\
\qquad\quad\quad \Phi(f \circ m) \\
\qquad [\,(m :\to \mu F^{\mathrm{e}})\, \Phi(f \circ m)\,] \\
\qquad = \quad -\ [\,\cdot\,]\text{-char-uniq}\ - \\
\qquad\quad \triangleright\, m :\to \mu F^{\mathrm{e}} \\
\qquad\quad \overline{\quad \Phi(f \circ m)} \\
\qquad\quad = \quad -\ [\,\cdot\,]\text{-char-exi}\ - \\
\qquad\quad\quad [\,\Phi\,] \circ \mathtt{in}_{F^{\triangleright}C}(f \circ m) \\
\qquad\quad = \quad -\ [\,\cdot\,]\text{-char-exi}\ - \\
\qquad\quad\quad [\,\Phi\,] \circ [\,(m :\to \mu F^{\mathrm{e}})\, \mathtt{in}_{F^{\triangleright}C}(f \circ m)\,] \circ \mathtt{in}_{F^{\triangleright}(\mu F^{\mathrm{e}})} m \\
\qquad [\,\Phi\,] \circ [\,(m :\to (\mu F^{\mathrm{e}}))\, \mathtt{in}_{F^{\triangleright}C}(f \circ m)\,] \\
\qquad = \quad -\ F^{\mathrm{e}}\text{-def}\ - \\
\qquad [\,\Phi\,] \circ F^{\mathrm{e}} f \\
\quad (\!|\,[\,\Phi\,]\,|\!)_{F^{\mathrm{e}}} \\
= \quad -\ (\!|\cdot|\!)^{\mathrm{m}}_{F}\text{-def}\ - \\
\quad (\!|\,\Phi\,|\!)^{\mathrm{m}}_{F}
\end{array}
$$

The following fact explains why we were able to obtain a simpler reduction in the covariant case (i.e., why we could use an initial algebra for F instead of $(F^{\wr})^{\mathrm{e}}$). Given a covariant functor F such that, for any object C, there exists a sum of $(F^{\wr})^{\triangleright}C$. Then, for any object C, the objects FC and $(F^{\wr})^{\mathrm{e}}C$ are isomorphic, and being an (initial) F-algebra is equivalent to being an (initial) $(F^{\wr})^{\mathrm{e}}$-algebra.

6 Conclusions and Future Work

This paper is a first report of an ongoing work, so we have several questions that we have to find answers to. It seems clear already, however, that the notion of initial Mendler-style algebras is meaningful and that

5 Mendler-style inductive types reduced to inductive types

5.1 Covariant case, easy

Let again F be a covariant functor, so $F^{\wr}$ is a difunctor. Let us assume that there exists an initial F-algebra $((\mu F, \mathtt{In}_F), (\!|\cdot|\!)_F)$. Define

$(\mu^{\mathrm{m}}F^{\wr}, \mathtt{In}^{\mathrm{m}}_{F^{\wr}})$-def

$$\mu^{\mathrm{m}}F^{\wr} = \mu F,\ \ \mathtt{In}^{\mathrm{m}}_{F^{\wr}} = (m :\to \mu F)\, \mathtt{In}_F \circ Fm$$

$(\!|\cdot|\!)^{\mathrm{m}}_{F^{\wr}}$-def

$$C : \mathrm{Obj},\ (\,\gamma :\to C \ \Rightarrow\ \Phi\gamma :\to C\,),\ (C,\Phi)\text{-}F^{\wr}\text{-extra} \ \Rightarrow\ (\!|\,\Phi\,|\!)^{\mathrm{m}}_{F^{\wr}} = (\!|\,\Phi\mathtt{id}_C\,|\!)_F$$

The pair $((\mu^{\mathrm{m}}F^{\wr}, \mathtt{In}^{\mathrm{m}}_{F^{\wr}}), (\!|\cdot|\!)^{\mathrm{m}}_{F^{\wr}})$ turns out to be an initial Mendler-style $F^{\wr}$-algebra.

Proof of $(\!|\cdot|\!)^{\mathrm{m}}_{F^{\wr}}$-char-exi:

$$
\begin{array}{l}
\triangleright\, C : \mathrm{Obj},\ (\,\gamma :\to C \ \Rightarrow\ \Phi\gamma :\to C\,),\ (C,\Phi)\text{-}F^{\wr}\text{-extra} \\
\hline
\triangleright\, m :\to \mu^{\mathrm{m}}F^{\wr} \\
\hline
\quad (\!|\,\Phi\,|\!)^{\mathrm{m}}_{F^{\wr}} \circ \mathtt{In}^{\mathrm{m}}_{F^{\wr}} m \\
= \quad -\ (\!|\cdot|\!)^{\mathrm{m}}_{F^{\wr}}\text{-def},\ (\mu^{\mathrm{m}}F^{\wr}, \mathtt{In}^{\mathrm{m}}_{F^{\wr}})\text{-def}\ - \\
\quad (\!|\,\Phi\mathtt{id}_C\,|\!)_F \circ \mathtt{In}_F \circ Fm \\
= \quad -\ (\!|\cdot|\!)_F\text{-char-exi}\ - \\
\quad \Phi\mathtt{id}_C \circ F(\!|\,\Phi\mathtt{id}_C\,|\!)_F \circ Fm \\
= \quad -\ (\!|\cdot|\!)^{\mathrm{m}}_{F^{\wr}}\text{-def}\ - \\
\quad \Phi\mathtt{id}_C \circ F(\!|\,\Phi\,|\!)_{F^{\wr}} \circ Fm \\
= \quad -\ F\ \text{cov functor}\ - \\
\quad \Phi\mathtt{id}_C \circ F((\!|\,\Phi\,|\!)_{F^{\wr}} \circ m) \\
= \quad -\ F^{\wr}\text{-def}\ - \\
\quad \Phi\mathtt{id}_C \circ F^{\wr}(\mathtt{id}_C, (\!|\,\Phi\,|\!)_{F^{\wr}} \circ m) \\
= \quad -\ \lhd\ - \\
\quad \Phi((\!|\,\Phi\,|\!)^{\mathrm{m}}_{F^{\wr}} \circ m) \circ F^{\wr}((\!|\,\Phi\,|\!)_{F^{\wr}} \circ m, \mathtt{id}_{\mathtt{dom}\,m}) \\
= \quad -\ F^{\wr}\text{-def}\ - \\
\quad \Phi((\!|\,\Phi\,|\!)^{\mathrm{m}}_{F^{\wr}} \circ m) \circ F\mathtt{id}_{\mathtt{dom}\,m} \\
= \quad -\ F\ \text{cov functor}\ - \\
\quad \Phi((\!|\,\Phi\,|\!)^{\mathrm{m}}_{F^{\wr}} \circ m)
\end{array}
$$

Proof of $(\!|\cdot|\!)^{\mathrm{m}}_{F^l}$-char-uniq:

$$
\begin{array}{|l}
\triangleright\, C:\mathrm{Obj},\ (\,\gamma:\to C\ \Rightarrow\ \Phi\gamma:\to C\,),\ (C,\Phi)\text{-}F^l\text{-extra}\\
\hline
\triangleright\, f:\mu^{\mathrm{m}}F^l\to C,\ (\,m:\to\mu^{\mathrm{m}}F^l\ \Rightarrow\ f\circ\mathtt{In}^{\mathrm{m}}_{F^l}m=\Phi(f\circ m)\,)\\
\hline
\quad f\\
= \qquad -\ (\!|\cdot|\!)_F\text{-char-uniq}\ -\\
\quad \begin{array}{|l}
f\circ\mathtt{In}_F\\
= \qquad -\ F\ \text{cov functor}\ -\\
f\circ\mathtt{In}_F\circ F\mathtt{id}_{\mu F}\\
= \qquad -\ (\mu^{\mathrm{m}}F^l,\mathtt{In}^{\mathrm{m}}_{F^l})\text{-def}\ -\\
f\circ\mathtt{In}^{\mathrm{m}}_{F^l}\mathtt{id}_{\mu^{\mathrm{m}}F^l}\\
= \qquad -\ \lhd\ -\\
f\circ\Phi f\\
= \qquad -\ F\ \text{cov functor}\ -\\
f\circ\Phi f\circ F\mathtt{id}_{\mu^{\mathrm{m}}F^l}\\
= \qquad -\ F^l\text{-def}\ -\\
f\circ\Phi f\circ F^l(f,\mathtt{id}_{\mu^{\mathrm{m}}F^l})\\
= \qquad -\ \lhd\ -\\
\Phi\mathtt{id}_C\circ F(\mathtt{id}_C,f)\\
= \qquad -\ F^l\text{-def}\ -\\
\Phi\mathtt{id}_C\circ Ff
\end{array}\\
\quad (\!|\,\Phi\mathtt{id}_C\,|\!)_F\\
= \qquad -\ (\!|\cdot|\!)^{\mathrm{m}}_{F^l}\text{-def}\ -\\
\quad (\!|\,\Phi\,|\!)^{\mathrm{m}}_{F^l}
\end{array}
$$

5.2 General (i.e., di-) case, harder

Let now F be a difunctor. Define a mapping $F^{\triangleright}$ which takes any object C to a family of objects indexed by morphisms with codomain C by letting $F^{\triangleright}C=(\gamma:\to C)\,F(\mathtt{dom}\,\gamma,\mathtt{dom}\,\gamma)$. Assume that, for any object C, there exists a coproduct $((\sum F^{\triangleright}C,\mathtt{in}_{F^{\triangleright}C}),[\,\cdot\,])$ of $F^{\triangleright}C$, i.e., that the following judgements hold:

$(\sum F^{\triangleright}\cdot,\mathtt{in}_{F^{\triangleright}\cdot})$**-prop**

$$
\begin{array}{l}
C:\mathrm{Obj}\\
\quad\Rightarrow\ \sum F^{\triangleright}C:\mathrm{Obj},(\,\gamma:\to C\ \Rightarrow\ \mathtt{in}_{F^{\triangleright}C}\gamma:\to\sum F^{\triangleright}C\,),\\
\qquad(\sum F^{\triangleright}C,\mathtt{in}_{F^{\triangleright}C})\text{-}C\text{-}F\text{-extra}
\end{array}
$$

$[\,\cdot\,]$**-char-exi**

$$
\begin{array}{l}
C:\mathrm{Obj},\ E:\mathrm{Obj},\ (\,\gamma:\to C\ \Rightarrow\ \Theta\gamma:\to E\,),\ (C,\Theta)\text{-}F\text{-extra}\\
\quad\Rightarrow\ [\,\Theta\,]:\sum F^{\triangleright}C\to E,\ (\,\gamma:\to C\ \Rightarrow\ [\,\Theta\,]\circ\mathtt{in}_{F^{\triangleright}C}\gamma=\Theta\gamma\,)
\end{array}
$$

$[\,\cdot\,]$**-char-uniq**

$$
\begin{array}{l}
C:\mathrm{Obj},\ E:\mathrm{Obj},\ (\,\gamma:\to C\ \Rightarrow\ \Theta\gamma:\to E\,),\ (C,\Theta)\text{-}F\text{-extra}\\
\quad\Rightarrow\ (\,f:\sum F^{\triangleright}C\to E,\ (\,\gamma:\to C\ \Rightarrow\ f\circ\mathtt{in}_{F^{\triangleright}C}\gamma=\Theta\gamma\,)\\
\qquad\Rightarrow\ f=[\,\Theta\,]\,)
\end{array}
$$

Define an object and morphism mapping F^{e} as follows:

F^{e}**-def**

$$
\begin{array}{l}
C:\mathrm{Obj}\ \Rightarrow\ F^{\mathrm{e}}C=\sum F^{\triangleright}C\\
\quad h:\to\ \Rightarrow\ F^{\mathrm{e}}h=[\,(\gamma:\to\mathtt{dom}\,h)\,\mathtt{in}_{F^{\triangleright}(\mathtt{cod}\,h)}(h\circ\gamma)\,]
\end{array}
$$

It is easy to see that F^{e} is a covariant functor.

Assume that there exists an initial F^{e}-algebra $((\mu F^{\mathrm{e}},\mathtt{In}_{F^{\mathrm{e}}}),(\!|\cdot|\!)_{F^{\mathrm{e}}})$. Define

$(\mu^{\mathrm{m}}F,\mathtt{In}^{\mathrm{m}}_F)$**-def**

$$\mu^{\mathrm{m}}F=\mu F^{\mathrm{e}},\ \mathtt{In}^{\mathrm{m}}_F=(m:\to\mu F^{\mathrm{e}})\,\mathtt{In}_{F^{\mathrm{e}}}\circ\mathtt{in}_{F^{\triangleright}(\mu F^{\mathrm{e}})}m$$

$(\!|\cdot|\!)^{\mathrm{m}}_F$**-def**

$$C:\mathrm{Obj},\ (\,\gamma:\to C\ \Rightarrow\ \Phi\gamma:\to C\,),\ (C,\Phi)\text{-}F\text{-extra}\ \Rightarrow\ (\!|\,\Phi\,|\!)^{\mathrm{m}}_F=(\!|\,[\,\Phi\,]\,|\!)_{F^{\mathrm{e}}}$$

This definitions warrant that the pair $((\mu^{\mathrm{m}}F,\mathtt{In}^{\mathrm{m}}_F),(\!|\cdot|\!)^{\mathrm{m}}_F)$ is an initial Mendler-style F-algebra.

Proof of $(\!|\cdot|\!)^{\mathrm{m}}_F$-char-exi:

$$
\begin{array}{|l}
\triangleright\, C:\mathrm{Obj},\ (\,\gamma:\to C\ \Rightarrow\ \Phi\gamma:\to C\,),\ (C,\Phi)\text{-}F\text{-extra}\\
\hline
\triangleright\, m:\to\mu^{\mathrm{m}}F\\
\hline
\quad (\!|\,\Phi\,|\!)^{\mathrm{m}}_F\circ\mathtt{In}^{\mathrm{m}}_F m\\
= \qquad -\ (\!|\cdot|\!)^{\mathrm{m}}_F\text{-def},\ (\mu^{\mathrm{m}}F,\mathtt{In}^{\mathrm{m}}_F)\text{-def}\ -\\
\quad (\!|\,[\,\Phi\,]\,|\!)_{F^{\mathrm{e}}}\circ\mathtt{In}_{F^{\mathrm{e}}}\circ\mathtt{in}_{F^{\triangleright}(\mu F^{\mathrm{e}})}m\\
= \qquad -\ (\!|\cdot|\!)_{F^{\mathrm{e}}}\text{-char-exi}\ -\\
\quad [\,\Phi\,]\circ F^{\mathrm{e}}(\!|\,[\,\Phi\,]\,|\!)_{F^{\mathrm{e}}}\circ\mathtt{in}_{F^{\triangleright}(\mu F^{\mathrm{e}})}m\\
= \qquad -\ (\!|\cdot|\!)^{\mathrm{m}}_F\text{-def}\ -\\
\quad [\,\Phi\,]\circ F^{\mathrm{e}}(\!|\,\Phi\,|\!)_F\circ\mathtt{in}_{F^{\triangleright}(\mu F^{\mathrm{e}})}m\\
= \qquad -\ F^{\mathrm{e}}\text{-def}\ -\\
\quad [\,\Phi\,]\circ[\,(m:\to\mu F^{\mathrm{e}})\,\mathtt{in}_{F^{\triangleright}C}((\!|\,\Phi\,|\!)^{\mathrm{m}}_F\circ m)\,]\circ\mathtt{in}_{F^{\triangleright}(\mu F^{\mathrm{e}})}m\\
= \qquad -\ [\,\cdot\,]\text{-char-exi}\ -\\
\quad [\,\Phi\,]\circ\mathtt{in}_{F^{\triangleright}C}((\!|\,\Phi\,|\!)^{\mathrm{m}}_F\circ m)\\
= \qquad -\ [\,\cdot\,]\text{-char-exi}\ -\\
\quad \Phi((\!|\,\Phi\,|\!)^{\mathrm{m}}_F\circ m)
\end{array}
$$