Uniqueness and Efficency Of Nash Equilibrium In A Family Of Randomly Generated Repeated Games (1),

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Abstract -This paper describes the results of an analysis of the Nash Equilibrium in randomly generated repeated games. We study two families of games: symmetric bi-matrix games G(A, B) with $B = A^T$ and non-symmetric bi-matrix games (the first includes the classical games of Prisoner Dilemma, Battle of the Sexes, and Chickens). We use pure strategies, implemented by automata of size two, and different strategy domination criteria.

We observe that, in this environment, the uniqueness and efficiency of Equilibria Outcomes is the typical result.

Keywords - Games Theory, Repeated Games, Nash-Equilibrium, Automata.

1.- Introduction

It is well known that the behavior of economic agents that interact repeatedly cannot be fully captured by simple static models. Repeated interaction allows agents to use punishments threats to enforce particular actions that would not have been taken in a static framework. In many economic examples the long term relationship consists of a sequence of repeated situations. This economic scheme can be modeled as a Repeated Game. A typical example of this type of interaction occurs when many firms interact in a market. Usually this interaction is not carried out in a single period but rather on a long term relationship. In this case some "cooperative" behavior is observed, even when there is no commitment among the actors. The possibility of future punishment yields cooperation in a non-cooperative environment.

Non-cooperative game theory provides equilibrium concepts in order to decide which type of behavior result stable. The Nash equilibrium (see [1]) is the most classical and generally accepted solution.

Usually several new equilibria appear when we compare a Repeated Game with a game corresponding to a single period. This result is known in the literature as the Folk Theorem. Having many equilibria is good in some sense. It allows supporting many outcomes (including some "cooperative outcomes"). But it is not so good in another sense. The multiplicity of equilibria makes the prediction of the actual path of play very difficult. In some typical examples there is a clear intuition on what should be the "natural" outcome of the game. For instance in the Prisoners Dilemma game, (C,C) should be the result (see section 3). Thus, we would like to see unique full cooperative outcomes as a result of only non-cooperative assumptions.

Several different approaches have been attempted towards this goal, but with almost no success in isolating the “cooperative” equilibrium. The most common approach focuses on refining

(1) We are very grateful to an anonymous referee for suggesting changes that have significantly improved the exposition.

the Nash solution (Subgameperfection [2] and other refinements, see [3] for details). It reduces the equilibrium strategies but not the equilibrium payoffs, bringing up again a full Folk Theorem. Thus extra rationality assumptions have to be included in the model, because some "non-cooperative outcomes" (D, D) in the Prisoners Dilemma game are hard to remove. Moreover sometimes the only remaining equilibrium is just the non-cooperative one (see [4])

Another approach includes some bounded rationality considerations on the complexity of the strategies or some preference for simple strategies and costs of strategies implementation. This research line was followed by [3],[5],[6] and others. A reduction of the Equilibrium outcomes was obtained but they failed to remove the non-cooperative one, and in some cases the full cooperative outcome was removed instead.

We will present a model with only non-cooperative assumptions. We will consider the Nash Equilibrium solution and the standard elimination of dominated strategies. We will only deal with strategies implemented by automata of size two. Even though this is certainly a very simple context, it is a fully non-cooperative situation. This class includes strategies (like tit-for-tat) that proved to be successful even in the presence of more complicated strategies (see the "Tournament approach" presented by [7]. and [8]).

In our model the non-cooperative outcome will be then eliminated in the Prisoners Dilemma Game (in this case the full cooperative outcome and a not fully cooperative one remain as possible outcomes).

Moreover, we will show that the typical result in a family of symmetric-games (including the Prisoners Dilemma Game) is uniqueness. It is also observed that we typically obtain efficient unique outcomes. We will also show that the uniqueness is still more common in general games if we remove the symmetry.

2.- Games in Normal Form and Repeated Games

A Game of strategies in normal form could be described by a 3-uple G= (N, A, u). N will be the set of players. We will deal with 2-players games, then N={1,2}. A is the action profile set, $A=A_1 \times A_2$. For each player $i \in N$, A_i is the set of available strategies. $u=(u_1,u_2)$ are the payoff utility vector, where $u_i : A \rightarrow R$ is the payoff function of player i.

A Nash Equilibrium (NE) is an action profile $a^* = (a^*_1, a^*_2)$ such as :
$\forall\ a_1 \in A_1,\ u_1(a^*) \geq u_1(a_1,a^*_2)$ and $\forall\ a_2 \in A_2,\ u_2(a^*) \geq u_2(a^*_1,a_2)$.

The (infinitely) Repeated Game consists of a sequence of repetitions of a one-shot game.

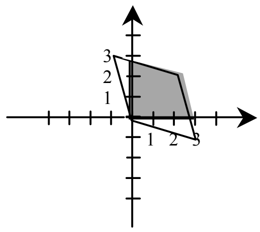
The sequence of outcomes will be evaluated with limit of the mean.

2.1- Prisoners dilemma and other Classical Games.

The Prisoners Dilemma (P=({1,2}, A, u)) is without a doubt the most studied game in Game Theory. This game can be described by the numeric example showed in Fig.1 (a).

	C	D
C	**2, 2**	**-1, 3**
D	**3,-1**	**0, 0**

(a)



(b)

Fig 1: Prisoners dilemma. (a) Payoffs matrix. (b) Possible equilibrium outcomes in the Repeated Game (Folk Theorem).

It is immediate to verify that the only Nash Equilibrium in the one shot game is (D, D) with payoff (0, 0). However the outcome (2,2) associated to the strategies (C, C) naturally results more appealing. When the game is repeated over time we have the Classical Folk Theorem, where the shaded area represent the equilibrium outcomes (Fig.1 (b)).

We will also consider the following classical games:

(a)

	C	D
C	**-4, -4**	**2, -1**
D	**-1, 2**	**0, 0**

(b)

	C	D
C	**3, 1**	**0, 0**
D	**0, 0**	**1, 3**

Fig.2 Classical Games. (a) Game of Chicken. (b) Battle of the Sexes.

3.- Use of Automata in Repeated Games.

The intuitive description of an automaton corresponds to think that an agent (a player) may have diverse states of mind under which it takes decisions. For example a player that plays the constant strategy **C** in the Infinitely Repeated Prisoners Dilemma Game has only one state of mind ("cooperative"). On the other hand if it plays a trigger strategy we could think that it has two states, an initial cooperative state and after any non-cooperation it move to a non-cooperative state where it remains for ever.

The C-tit for tat (C-tft) strategy and D-tit for tat (D-tft) also correspond to individuals with two states of mind.

The following is a formal description of a (*full*) automaton for player i. This will be denoted by a 4-uple :

$M_i = (M_i, m_i^0, B_i, T_i)$ with $M_i \subset N$ (N is the set of natural numbers), M_i is the set of states, m_i^0 is the initial state, B_i: $M_i \rightarrow A_i$ is the behavior function and T_i: M_i x $A \rightarrow M_i$ is the transition function (where $A=\Pi A_i$ is the actions profiles set). The transition function indicates how the automaton changes of states. When the transition function is restricted to $T_i: M_i \times A_{-i} \longrightarrow M_i$, where for j,i $\in$N and i $\neq$ j: $A_{-i}=\Pi A_j$,the automaton M_i will be called *Exact.*

The automaton will be said finite if M_i is a finite set.

Given a strategy we could define a full automaton that implements it and given a full automaton we could construct the associate strategy. The construction of an automaton starting from the strategy could be done in several different forms and we could give automata of different sizes associates to the same strategy. However there exists an automaton of minimal size that implements a given strategy.

A strategy in an extensive-form game typically indicates the actions to be taken by a given player in all information sets (in which he/she plays). This is done so that when studying refinements a player can tremble in a previous information set, and still have his/her strategy prescribe what to do in subsequent moves. Thus, each *exact automata* don not fully capture the notion of strategy. However to the effects of studying Nash Equilibria it won't make any difference considering full or exact automata. We use only exact automata.

In Figure 3 we show examples of both types of automata:

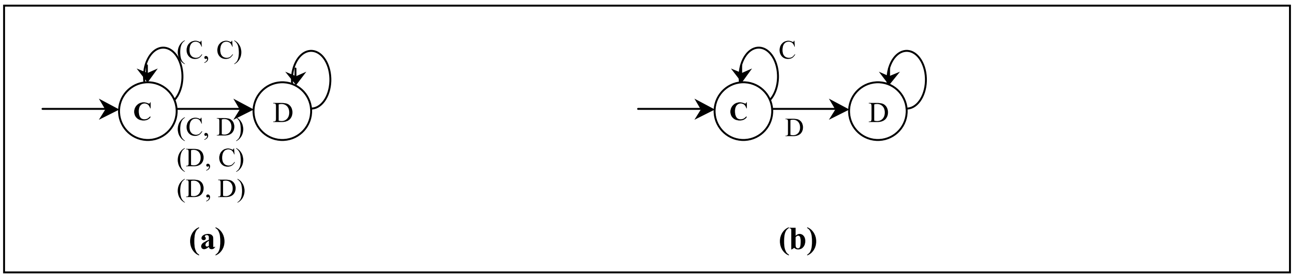


Fig 3.(a) Full automaton. Trigger strategy (with punishments to deviations of any player), (b) Exact automaton. Trigger strategy (with punishments to deviations of the opponent)

Here the behavior functions are indicated in the circles of the corresponding states and the arches between states indicate the transitions. The initial states are the first to the left and arches without labeling includes all the options not considered by the labeled arches.

4.- Folk Theorems for games played by two states automata with elimination of Dominated Strategies.

There are many articles in the literature attempting to give priority to the preference for simpler strategies (see [5], [6], [9], [10], etc).

We will deal with two player games with two pure strategies for each player, and we will restrict to strategies implemented by automata of sizes at most two. We will consider strategic domination.

A strategy f_i of player i dominates another strategy g_i of player i, in a S*trong way* (SDom) if the outcomes obtained by player i by playing f_i are strictly better than those obtained by playing g_i against any strategy of the opponent (f_i SDom g_i iff $u_i(f_i, r_j)>u_i(g_i, r_j)$ for all strategy r_j of player $j\neq i$).

If the inequality is not strict we have W*eak domination* (WDom), and if we also require that for at least one strategy of the opponent the inequality be strict we have *regular domination* (RDom).

5.- An statistic study of Nash Equilibrium in games played by two states automata

5.1. Classical Games

For the Prisoners Dilemma under the above mentioned assumptions we have:

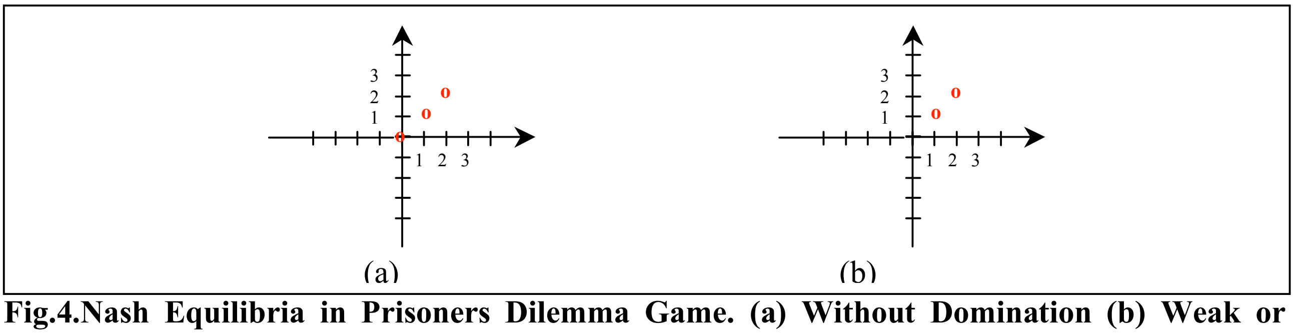


Fig.4.Nash Equilibria in Prisoners Dilemma Game. (a) Without Domination (b) Weak or Regular Domination.

The effect of domination in classical games was very weak. Only in the Prisoner Dilemma game we eliminated the non cooperative outcome (0, 0) but we still had two remaining outcomes, (1, 1) and (2, 2). In the others games there was no reductions in the number of outcomes.

5.2. Symmetrical Random Games

We generated 50 random symmetric bi-matrix games G(A, B) with $B = A^T$. It includes the classical games of Prisoner Dilemma, Battle of the Sexes, and Chickens. The entries were integer numbers in the range [-9,9]. We obtained the following results:

GAME	WITHOUT DOMINATION						
	#NEO	#Equilibrium Strategies					
1	1	35					
2	2	79	12				
3	3	79	14	4			
4	1	79					
5	3	79	52	7			
6	4	79	8	1	1		
7	1	39					
8	1	210					
9	3	79					
10	3	79	52	4			
11	3	79	14	4			
12	1	35					
13	3	57	57	52			
14	1	79					
15	3	27	27	52			
16	1	35					
17	1	79					
18	4	18	18	15	23		
19	3	79	14	4			
20	2	79	12				
21	6	30	30	24	24	54	9
22	1	35					
23	3	26	2	4			
24	4	34	34	52	7		
25	3	79	35	5			

GAME	WITHOUT DOMINATION (cont.)									
	#NEO	#Equilibrium strategies								
26	9	79	14	14	5	15	15	8	8	26
27	2	79	12							
28	3	35	79	48						
29	2	79	12							
30	4	32	57	57	52					
31	3	14	79	14						
32	2	79	12							
33	7	79	34	34	15	15	4	11		
34	3	79	52	41						
35	4	57	57	52	7					
36	1	210								
37	5	4	11	11	79	52				
38	3	10	4	1						
39	2	10	8							
40	2	4	12							
41	4	79	11	11	4					
42	3	41	79	52						
43	5	79	7	7	14	14				
44	3	4	2	4						
45	3	79	14	4						
46	6	26	26	21	21	8	54			
47	3	27	27	52						
48	2	79	12							
49	4	27	27	52	1					
50	1	39								

Table 1: Number of equilibrium outcomes without domination.

40	0	2	4	12					50	1	1						50	1	1					
41	0	4	79	11	11	4			10	4	59	4	4	4			10	4	59	4	4	4		
42	0	3	41	79	52				0	3	41	79	52				0	3	41	79	52			
43	0	5	79	7	7	14	14		49	1	2						50	1	1					
44	0	3	4	2	4				30	3	4	2	4				34	3	4	2	1			
45	0	3	79	14	4				14	2	71	14					14	2	71	14				
46	0	6	26	26	21	21	8	54	38	1						40	40	1						30
47	0	3	27	27	52				0	3	27	27	52				0	3	27	27	52			
48	2	2	79	12					23	2	41	9					24	2	41	7				
49	0	4	27	27	52	1			6	3	20	20	52				6	3	20	20	52			
50	5	1	39						46	1	9						50	1	1					

Table 2:Number of equilibrium outcomes with different domination.

GAME	STRONG DOMINATION			REGULAR DOMINATION			WEAK DOMINATION		
	#El.	#NEO	#Equilibrium Strategies	#El.	#NEO	#Equilibrium Strategies	#El.	#NEO	#Equilibrium Strategies

Table 2:Number of equilibrium outcomes with different domination (Cont.).

In the Table 1 we show the number of equilibrium outcomes with the different domination and in the Table 2 without domination. In these tables, **#NEO** is the number of different Nash Equilibrium Outcomes in each randomly generated game; **#Equilibrium Strategies** is the number strategies leading to each equilibrium outcome in an decreasing order (for instance: in the game 2, there are 2 payoffs corresponding to different equilibrium strategies; 79 strategies give the first outcome and 12 the second. This game has 91 equilibrium strategies with 2 different equilibrium outcomes); **#El.** is the number of strategies eliminated under each domination criterion.

GAME	#EO	EQUILIBRIA PAYOFF																	
1	1	(6,6)	**ES**																
2	2	(7,7)	**ES**	(5.50,5.50)	**S**														
3	3	(4,4)	**ES**	(0.00,0.00)	**S**	(-4,-4)	**S***												
4	1	(5,5)	**ES**																
5	3	(0.00,0.00)	**ES**	(-3.50,-3.50)	**S**	(-7,-7)	**S**												
6	4	(1,1)	**ES**	(-3.50,-3.50)	**S**	(-4,-5)	*	(-5,-4)	*										
7	1	(4,4)	**ES**																
8	1	(-3,-3)	**ES**																
9	3	(4,4)	**ES**	(3,3)	**S***	(2,2)	**S***												
10	3	(2,2)	**ES**	(-0.50,-0.50)	**S**	(-3,-3)	**S***												
11	3	(0.00,0.00)	**ES**	(-1.50,-1.50)	**S**	(-3,-3)	**S***												
12	1	(4,4)	**ES**																
13	3	(7,6)	**E**	(6,7)	**E**	(6.50,6.50)	**ES**												
14	1	(6,6)	**ES**																
15	3	(-0.50,-0.50)	**ES**	(-5,4)	**E**	(4,-5)	**E**												
16	1	(4,4)	**ES**																
17	1	(3,3)	**ES**																
18	4	(-6,2)	**E**	(2,-6)	**E**	(-2,-2)	**S**	(1,1)	**ES**										
19	3	(6,6)	**ES**	(1.50,1.50)	**S**	(-3,-3)	**S**												
20	2	(2,2)	**ES**	(-2,-2)	**S***														
21	6	(2,-8)	**E***	(-8,2)	**E***	(-3,-8)	*	(-8,-3)	*	(-8,-8)	**S***	(-3,-3)	**ES**						
22	1	(6,6)	**ES**																
23	3	(3,3)	**ES**	(0.50,0.50)	**S**	(-1,-1)	**S***												
24	4	(7,1)	**E**	(1,7)	**E**	(4,4)	**ES**	(3,3)	**S**										
25	3	(4,4)	**ES**	(2,2)	**S**	(0.00,0.00)	**S**												
26	9	(4,4)	**ES**	(-7,4)	*	(4,-7)	*	(-7,-7)	**S***	(4,-1.50)	*	(-1.50,4)	*	(-1.50,-7)	*	(-7,-1.50)	*	(-1.50,-1.50)	
27	2	(3,3)	**ES**	(0.00,0.00)	**S**														
28	3	(-2,-2)	**S**	(-1,-1)	**ES**	(-1.50,-1.50)	**S**												
29	2	(1,1)	**ES**	(-1,-1)	**S**														
30	4	(4,4)	**ES**	(0.00,4)		(4,0.00)		(2,2)	**S**										
31	3	(4,4)	**S**	(7,7)	**ES**	(5.50,5.50)	**S**												
32	2	(4,4)	**ES**	(0.00,0.00)	**S**														
33	7	(6,6)	**ES**	(6,-1)	*	(-1,6)	*	(2.50,6)	*	(6,2.50)	*	(-1,-1)	**S***	(2.50,2.50)	**S***				
34	3	(5,5)	**ES**	(4,4)	**S**	(3,3)	**S**												
35	4	(7,5)	**E**	(5,7)	**E**	(6,6)	**ES**	(5,5)	**S**										
36	1	(6,6)	**ES**																
37	5	(-7,-7)	**S***	(-6,-7)	*	(-7,-6)	*	(-5,-5)	**ES**	(-6,-6)	**S**								
38	3	(-2,-2)	**ES**	(-7,-7)	**S***	(-6,-6)	**S**												
39	2	(0.50,0.50)	**ES**	(0.00,0.00)	**S**														
40	2	(-1,-1)	**ES**	(-3,-3)	**S***														
41	4	(7,7)	**ES**	(4,-3)		(-3,4)		(0.00,0.00)	**S**										
42	3	(3,3)	**S**	(5,5)	**ES**	(4,4)	**S**												
43	5	(1,1)	**ES**	(1,-2)	*	(-2,1)	*	(-2,-2)	**S***	(0.50,0.50)	**S***								
44	3	(3,3)	**ES**	(2.50,2.50)	**S**	(1,1)	**S**												
45	3	(7,7)	**ES**	(1,1)	**S**	(-5,-5)	**S***												
46	6	(5,0.00)	**E***	(0.00,5)	**E***	(2.50,0.00)	*	(0.00,2.50)	*	(0.00,0.00)	**S***	(2.50,2.50)	**ES**						

Table 3: Equilibrium payoffs.

GAME	#EO	EQUILIBRIA PAYOFF(cont)																	
11	3	(0.00,0.00)	**ES**	(-1.50,-1.50)	**S**	(-3,-3)	**S***												
12	1	(4,4)	**ES**																
13	3	(7,6)	**E**	(6,7)	**E**	(6.50,6.50)	**ES**												
14	1	(6,6)	**ES**																
15	3	(-0.50,-0.50)	**ES**	(-5,4)	**E**	(4,-5)	**E**												
16	1	(4,4)	**ES**																
17	1	(3,3)	**ES**																
18	4	(-6,2)	**E**	(2,-6)	**E**	(-2,-2)	**S**	(1,1)	**ES**										
19	3	(6,6)	**ES**	(1.50,1.50)	**S**	(-3,-3)	**S**												
20	2	(2,2)	**ES**	(-2,-2)	**S***														
21	6	(2,-8)	**E***	(-8,2)	**E***	(-3,-8)	*	(-8,-3)	*	(-8,-8)	**S***	(-3,-3)	**ES**						
22	1	(6,6)	**ES**																
23	3	(3,3)	**ES**	(0.50,0.50)	**S**	(-1,-1)	**S***												
24	4	(7,1)	**E**	(1,7)	**E**	(4,4)	**ES**	(3,3)	**S**										
25	3	(4,4)	**ES**	(2,2)	**S**	(0.00,0.00)	**S**												
26	9	(4,4)	**ES**	(-7,4)	*	(4,-7)	*	(-7,-7)	**S***	(4,-1.50)	*	(-1.50,4)	*	(-1.50,-7)	*	(-7,-1.50)	*	(-1.50,-1.50)	**S**
27	2	(3,3)	**ES**	(0.00,0.00)	**S**														
28	3	(-2,-2)	**S**	(-1,-1)	**ES**	(-1.50,-1.50)	**S**												
29	2	(1,1)	**ES**	(-1,-1)	**S**														
30	4	(4,4)	**ES**	(0.00,4)		(4,0.00)		(2,2)	**S**										
31	3	(4,4)	**S**	(7,7)	**ES**	(5.50,5.50)	**S**												
32	2	(4,4)	**ES**	(0.00,0.00)	**S**														
33	7	(6,6)	**ES**	(6,-1)	*	(-1,6)	*	(2.50,6)	*	(6,2.50)	*	(-1,-1)	**S***	(2.50,2.50)	**S***				
34	3	(5,5)	**ES**	(4,4)	**S**	(3,3)	**S**												
35	4	(7,5)	**E**	(5,7)	**E**	(6,6)	**ES**	(5,5)	**S**										
36	1	(6,6)	**ES**																
37	5	(-7,-7)	**S***	(-6,-7)	*	(-7,-6)	*	(-5,-5)	**ES**	(-6,-6)	**S**								
38	3	(-2,-2)	**ES**	(-7,-7)	**S***	(-6,-6)	**S**												
39	2	(0.50,0.50)	**ES**	(0.00,0.00)	**S**														
40	2	(-1,-1)	**ES**	(-3,-3)	**S***														
41	4	(7,7)	**ES**	(4,-3)		(-3,4)		(0.00,0.00)	**S**										
42	3	(3,3)	**S**	(5,5)	**ES**	(4,4)	**S**												
43	5	(1,1)	**ES**	(1,-2)	*	(-2,1)	*	(-2,-2)	**S***	(0.50,0.50)	**S***								
44	3	(3,3)	**ES**	(2.50,2.50)	**S**	(1,1)	**S**												
45	3	(7,7)	**ES**	(1,1)	**S**	(-5,-5)	**S***												
46	6	(5,0.00)	**E***	(0.00,5)	**E***	(2.50,0.00)	*	(0.00,2.50)	*	(0.00,0.00)	**S***	(2.50,2.50)	**ES**						

Table 3:Equilibrium payoffs. (Cont.)

In the Table 3 we show the equilibrium payoffs. **#NO** is the number of different outcomes. The references that appear together with the payoffs, indicate: **S** that the equilibrium is symmetric, **E** that the equilibrium is efficient (Pareto Optimal), * that the equilibrium disappears eliminating regular or weakly dominated strategies.

The following Picture shows the percentages of the number of equilibrium payoff with and without domination:

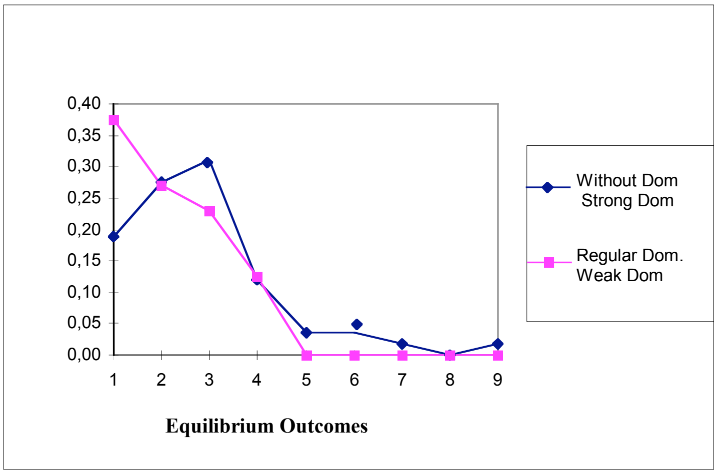


Fig.4: Symmetrical Random Games, percentages of the number of equilibrium payoffs with and without domination

We can observe that the typical number of equilibria decreases from 3 to 1 with the use of Weak or Regular domination. We also observe that eliminated equilibria are not the Efficient

Symmetrical (ES). The domination reduced the Symmetrical-Not Efficient Equilibria in 13% of the cases, Non Symmetrical- Efficient in 3% and Non Symmetrical-Non Efficient in 15%.

This certainly contrast with the multiplicity observed in these cases for the Classical Games in the previous section.

5.3. Non Symmetrical Random Games

Then we generated 50 random games (not necessary symmetric), under the same conditions of the previous section. We obtained the following results:

Game	WITHOUT DOMINATION							STRONG DOMINATION								REGULAR DOMINATION								WEAK DOMINATION							
	#NEO	#Equilibrium Estrategies						#Elim.	#NEO	#Equilibrium Estrategies						#Elim.	#NEO	#Equilibrium Estrategies						#Elim.	#NEO	#Equilibrium Estrategies					
1	3	79	45	41				0	3	79	45	41				0	3	79	45	41				0	3	79	45	41			
2	1	27						8	1	27						34	1	27						37	1	19					
3	1	18						8	1	18						46	1	5						50	1	1					
4	2	24	1					1	2	24	1					47	1	6						50	1	1					
5	4	79	18	15	6			0	4	79	18	15	6			47	1	4						50	1	1					
6	3	79	45	18				0	3	79	45	18				3	3	79	45	15				3	3	79	45	15			
7	6	45	27	27	22	19	8	0	6	45	27	27	22	19	8	38	1	34						40	1	25					
8	3	79	12	1				4	3	79	12	1				23	2	41	9					24	2	41	7				
9	1	24						11	1	24						44	1	12						50	1	1					
10	2	19	10					6	2	19	10					47	1	6						50	1	1					
11	2	46	11					0	2	46	11					6	2	32	10					6	2	32	10				
12	1	79						12	1	79						39	1	35						40	1	30					
13	1	8						4	1	8						49	1	2						50	1	1					
14	2	18	16					5	2	18	16					46	1	8						50	1	1					
15	4	79	23	20	13			0	4	79	23	20	13			48	1	3						50	1	1					
16	1	18						8	1	18						48	1	3						50	1	1					
17	1	79						8	1	79						37	1	45						40	1	30					
18	2	51	46					8	2	51	46					42	2	6	6					46	2	2	2				
19	2	79	10					8	2	79	10					37	1	45						40	1	30					
20	1	3						0	1	3						5	1	2						5	1	2					
21	3	79	12	1				4	3	79	12	1				23	2	41	9					24	2	41	7				
22	1	27						5	1	27						34	1	25						37	1	19					
23	5	18	16	11	8	4		0	5	18	16	11	8	4		46	2	4	4					50	1	1					
24	2	14	5					4	2	14	5					49	1	2						50	1	1					
25	1	46						0	1	46						4	1	33						4	1	33					
26	2	57	21					4	2	57	21					24	2	41	16					25	2	41	12				
27	1	19						0	1	19						3	1	13						3	1	13					
28	1	46						0	1	46						4	1	33						4	1	33					
29	2	51	19					8	2	51	19					40	2	3	2					42	2	3	2				
30	2	79	10					8	2	79	10					37	1	45						40	1	30					
31	2	16	11					0	2	16	11					7	2	11	10					7	2	11	10				
32	3	57	45	27				0	3	57	45	27				19	2	52	35					19	2	52	35				
33	3	24	19	8				4	3	24	19	8				36	2	14	12					50	1	1					
34	3	57	34	24				0	3	57	34	24				50	1	1						50	1	1					
35	1	14						8	1	14						49	1	2						50	1	1					
36	1	18						8	1	18						48	1	3						50	1	1					
37	2	79	12					9	2	79	12					25	2	41	6					26	2	41	4				
38	3	45	27	27				0	3	45	27	27				0	3	45	27	27				0	3	45	27	27			
39	4	79	11	7	4			0	4	79	11	7	4			18	3	37	11	6				18	3	37	11	6			
40	1	20						8	1	20						48	1	4						50	1	1					
41	1	51						8	1	51						48	1	4						50	1	1					
42	3	18	11	5				0	3	18	11	5				8	3	15	5	4				8	3	15	5	4			
43	1	79						17	1	79						43	1	14						50	1	1					
44	2	46	24					0	2	46	24					41	2	6	2					42	2	3	2				
45	2	16	5					0	2	16	5					7	2	11	4					7	2	11	4				
46	2	46	5					0	2	46	5					6	2	32	4					6	2	32	4				
47	1	79						8	1	79						37	1	45						40	1	30					
48	2	79	21					2	2	79	21					23	2	41	18					24	2	41	14				
49	1	46						0	1	46						7	1	27						7	1	27					
50	3	79	24	11				0	3	79	24	11				12	3	48	21	5				12	3	48	21	5			

Table 4:Number of equilibrium outcomes with and without different domination.

GAME	#EO	Equilibrium Payoff												
1	3	(6, 8)	E	(5, 6.50)		(4, 5)								
2	1	(-5, 3)	E											
3	1	(2, -8)	E											
4	2	(6, 3)	E	(0, 3)	*									
5	4	(3, -2)	E	(-2, -2)	*	(0.50, -2)	*	(0, -3)	*					
6	3	(7, 6)	E	(7, 2)		(7, -2)								
7	6	(3.50, -1.50)	E	(7, -2)	E*	(0, -1)	E*	(3.50, -2)	*	(0, -1.50)	*	(0, -2)	*	
8	3	(7, 7)	E	(0.50, -1)		(-2, -4)	*							
9	1	(-2, 7)	E											
10	2	(2, 0)		(2, 1.50)	*									
11	2	(7, 1)	E	(3, -5)										
12	1	(7, 0)	E											
13	1	(0, -2)	E											
14	2	(4, -1)	E	(9, -2)	E*									
15	4	(8, 0)	E	(0, -2)	*	(0, 0)	*	(4, -1)	*					

Table 5: Equilibrium payoffs.

GAME	#EO	Equilibrium Payoff (cont.)									
16	1	(-4, 7)	E								
17	1	(6, 1)	E								
18	2	(9, 6)	E	(6, 9)	E						
19	2	(6, 9)	E	(4, 4)	*						
20	1	(3, 1)	E								
21	3	(7, 9)	E	(1.50, 3.50)		(-3, -1)	*				
22	1	(-2, -6)	E								
23	5	(7, -9)	E*	(1.50, -9)	*	(-1, -4)	*	(1, -4)	E	(-4, -9)	*
24	2	(4, 1)		(4, 4.50)	E*						
25	1	(5, -3)	E								
26	2	(4, 0)	E	(0, 1)	E						
27	1	(2, 7)	E								
28	1	(5, 5)	E								
29	2	(4, 8)	E	(6, 5)	E						
30	2	(3, 2)	E	(1, 0)	*						
31	2	(8, -1)	E	(6, -4)							
32	3	(6, 8)	E	(0, 8.50)	E	(-6, 9)	E*				
33	3	(-8, 2)	E	(-8, -1)	*	(-8, -4)	*				
34	3	(2, 3)	E	(0, 6)	E*	(1, 4.50)	E*				
35	1	(3, -6)									
36	1	(0, 6)	E								
37	2	(9, 9)	E	(1.50, 6.50)							
38	3	(3, -2.50)	E	(8, -4)	E	(-2, -1)	E				
39	4	(3, 7)	E	(0.50, -0.50)		(3, -1)		(-2, -8)	*		
40	1	(-2, 2)									
41	1	(5, 4)	E								
42	3	(-8, 8)	E	(-1, -2)		(2, 1)	E				
43	1	(9, 9)	E								
44	2	(9, 0)	E	(-2, 1)	E						
45	2	(4, 0)	E	(-4, -4)							
46	2	(0, 4)	E	(-6, -2)							
47	1	(6, 9)	E								
48	2	(4, 7)	E	(-1.50,1.50)							
49	1	(1, 7)	E								
50	3	(8, 6))	E	(5, 5)		(2, 4)					

Table 5:Equilibrium payoffs (cont.)

The references are the same of table 1-2 and 3 respectively.

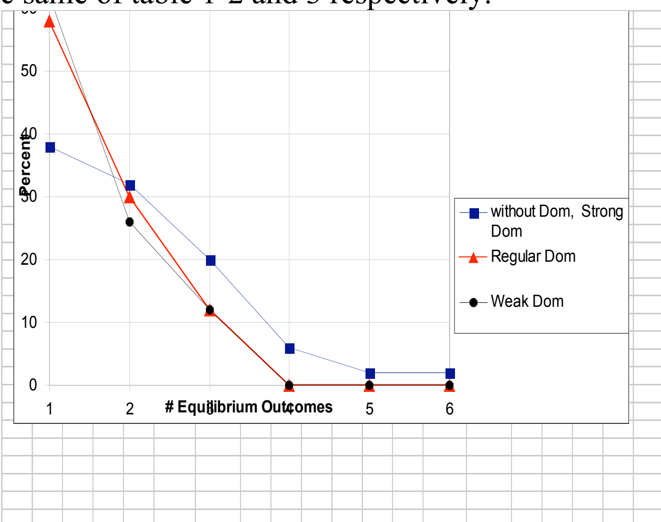


Fig.5: Non Symmetrical Random Games, percentages of equilibrium outcomes with and without different domination

Without the symmetrical restriction on the stage game, we found an enforcement of the equilibrium uniqueness in the repeated game. We also noted that under Strong domination no outcome is eliminated (only some strategies), Weak and Regular domination strength the uniqueness.

We have observed in this case, that the domination reduced the amount of Equilibrium in 27.8%, Not Efficient Equilibrium in 50% and Efficient in 12%:

Initially in the 38% of the cases the amount of equilibrium was one. After elimination of dominated strategies, the uniqueness was the result in 58% of the cases independently of the type of domination used.

8.- Concluding Remarks and Open Questions:

The use of automata of size 2 and strategy domination gives a full non-cooperative environment where the uniqueness and efficiency of the Equilibrium Outcomes is the typical result. It certainly contrast with the corresponding results in classical games and moreover with the unrestricted Folk Theorems.

For this study we used software that deals with automata of size 2. It would be interesting to relax the condition of the automata size, but it should be kept in mind that in this case several implementation difficulties can appear because of the large number of automata that one will have to manage. It would also be of interest to consider refinements of Nash Equilibria.

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