

# Trace equivalence and bisimilarity

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# Trace

## Definition

Let  $T = \langle S, N, \downarrow, \longrightarrow \rangle$  be a labelled transition system. The set of **traces**  $\text{Tr}(s)$ , for  $s \in S$  is the minimal set satisfying

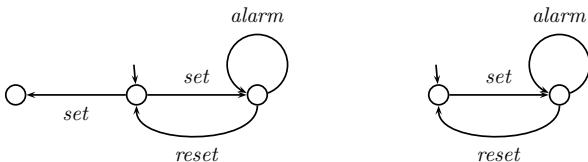
- (1)  $\epsilon \in \text{Tr}(s)$
- (2)  $\checkmark \in \text{Tr}(s) \Leftrightarrow s \in \downarrow$
- (3)  $a\sigma \in \text{Tr}(s) \Rightarrow \langle \exists s' : s' \in S : s \xrightarrow{a} s' \wedge \sigma \in \text{Tr}(s') \rangle$

# Trace equivalence

## Definition

Two states  $s, r$  are **trace equivalent** iff  $\text{Tr}(s) = \text{Tr}(r)$

## Example



**Trace equivalence** applies when one can neither interact with a system, nor distinguish a slow system from one that has come to a stand still.

# Simulation

the quest for a **behavioural equality**:  
able to identify states that cannot be distinguished by any **realistic**  
form of observation

## Simulation

A state  $q$  **simulates** another state  $p$  if every transition from  $q$  is corresponded by a transition from  $p$  and this capacity is kept along the whole life of the system to which state space  $q$  belongs to.

# Simulation

## Definition

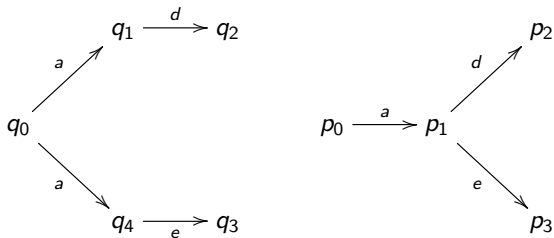
Given  $\langle S_1, N, \downarrow_1, \longrightarrow_1 \rangle$  and  $\langle S_2, N, \downarrow_2, \longrightarrow_2 \rangle$  over  $N$ , relation  $R \subseteq S_1 \times S_2$  is a **simulation** iff, for all  $\langle p, q \rangle \in R$  and  $a \in N$ ,

$$(1) \quad p \downarrow_1 \Rightarrow q \downarrow_2$$

$$(2) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$



# Example



$$q_0 \lesssim p_0 \quad \text{cf.} \quad \{\langle q_0, p_0 \rangle, \langle q_1, p_1 \rangle, \langle q_4, p_1 \rangle, \langle q_2, p_2 \rangle, \langle q_3, p_3 \rangle\}$$

# Similarity

## Definition

$$p \lesssim q \equiv \langle \exists R :: R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

## Lemma

The similarity relation is a preorder  
(ie, reflexive and transitive)

# Bisimulation

## Definition

Given  $\langle S_1, N, \downarrow_1, \longrightarrow_1 \rangle$  and  $\langle S_2, N, \downarrow_2, \longrightarrow_2 \rangle$  over  $N$ , relation  $R \subseteq S_1 \times S_2$  is a **bisimulation** iff both  $R$  and its converse  $R^\circ$  are simulations.

I.e., whenever  $\langle p, q \rangle \in R$  and  $a \in N$ ,

$$(1) \quad p \downarrow_1 \Leftrightarrow q \downarrow_2$$

$$(2.1) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$

$$(2.1) \quad q \xrightarrow{a}_2 q' \Rightarrow \langle \exists p' : p' \in S_1 : p \xrightarrow{a}_1 p' \wedge \langle p', q' \rangle \in R \rangle$$



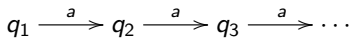
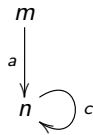
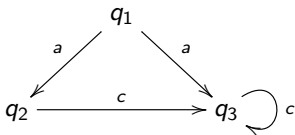
# Bisimulation

## The Game characterization

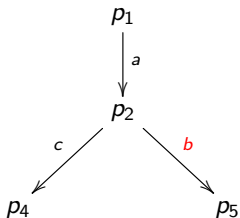
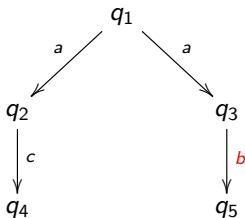
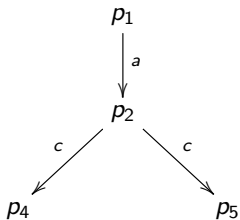
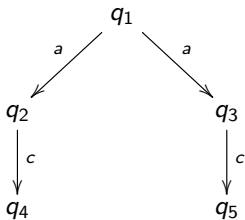
Two players  $R$  and  $I$  discuss whether the transition structures are mutually corresponding

- $R$  starts by choosing a transition
- $I$  replies trying to match it
- if  $I$  succeeds,  $R$  plays again
- $R$  wins if  $I$  fails to find a corresponding match
- $I$  wins if it replies to all moves from  $R$  and the game is in a configuration where all states have been visited or  $R$  can't move further. In this case is said that  $I$  has a **wining strategy**

# Examples



# Examples



# Bisimilarity

## Definition

$$p \sim q \equiv \langle \exists R :: R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

## Lemma

1. The identity relation  $\text{id}$  is a bisimulation
2. The empty relation  $\perp$  is a bisimulation
3. The converse  $R^\circ$  of a bisimulation is a bisimulation
4. The composition  $S \cdot R$  of two bisimulations  $S$  and  $R$  is a bisimulation
5. The  $\bigcup_{i \in I} R_i$  of a family of bisimulations  $\{R_i \mid i \in I\}$  is a bisimulation

# Bisimilarity

## Lemma

The bisimilarity relation is an equivalence relation  
(ie, reflexive, symmetric and transitive)

## Lemma

The class of all bisimulations between two LTS has the structure of a **complete lattice**, ordered by set inclusion, whose top is the **bisimilarity** relation  $\sim$ .

# Bisimilarity

## Lemma

In a **deterministic** labelled transition system, two states are bisimilar iff they are trace equivalent, i.e.,

$$s \sim s' \Leftrightarrow \text{Tr}(s) = \text{Tr}(s')$$

Hint: define a relation  $R$  as

$$\langle x, y \rangle \in R \Leftrightarrow \text{Tr}(x) = \text{Tr}(y)$$

and show  $R$  is a bisimulation.

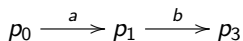
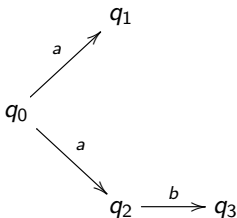
# Bisimilarity

## Warning

The bisimilarity relation  $\sim$  is not the symmetric closure of  $\lesssim$

## Example

$$q_0 \lesssim p_0, p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$



# Notes

Similarity as the greatest simulation

$$\lesssim \triangleq \bigcup \{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

$$\sim \triangleq \bigcup \{S \mid S \text{ is a bisimulation}\}$$