Bisimulation

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Simulation

the quest for a behavioural equality:
able to identify states that cannot be distinguished by any realistic form of observation

Simulation

A state q simulates another state p if every transition from q is corresponded by a transition from p and this capacity is kept along the whole life of the system to which state space q belongs to.

Simulation

Definition

Given $\langle S_1, N, \downarrow_1, \longrightarrow_1 \rangle$ and $\langle S_2, N, \downarrow_2, \longrightarrow_2 \rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a simulation iff, for all $\langle p, q \rangle \in R$ and $a \in N$,

$$(1) \quad p \downarrow_1 \; \Rightarrow q \downarrow_2$$

$$(2) \quad p \xrightarrow{a}_1 p' \; \Rightarrow \; \langle \exists\, q' \;:\; q' \in S_2 : \; q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$

$$\begin{array}{ccc} p \;\; R \;\; q & & q \\ \big\downarrow a & \Rightarrow & \big\downarrow a \\ p' & & p' \;\; R \;\; q' \end{array}$$

Example

$$
\begin{tikzcd}
 & q_1 \arrow[r, "d"] & q_2 \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_4 \arrow[r, "e"'] & q_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & p_2 \\
p_0 \arrow[r, "a"] & p_1 \arrow[ur, "d"] \arrow[dr, "e"'] & \\
 & & p_3
\end{tikzcd}
$$

$q_0 \lesssim p_0$ cf. $\{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$

Similarity

Definition

$$p \lesssim q \;\Leftrightarrow\; \langle \exists\, R \;::\; R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

Lemma

The similarity relation is a preorder
(ie, reflexive and transitive)

Bisimulation

Definition

Given $\langle S_1, N, \downarrow_1, \longrightarrow_1 \rangle$ and $\langle S_2, N, \downarrow_2, \longrightarrow_2 \rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a bisimulation iff both R and its converse R° are simulations.
I.e., whenever $\langle p, q \rangle \in R$ and $a \in N$,

(1) $p \downarrow_1 \;\Leftrightarrow q \downarrow_2$

(2.1) $p \xrightarrow{a}_1 p' \;\Rightarrow\; \langle \exists\, q' \;:\; q' \in S_2 : \; q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$

(2.1) $q \xrightarrow{a}_2 q' \;\Rightarrow\; \langle \exists\, p' \;:\; p' \in S_1 : \; p \xrightarrow{a}_1 p' \wedge \langle p', q' \rangle \in R \rangle$

Bisimulation

The Game characterization

Two players R and I discuss whether the transition structures are mutually corresponding

- R starts by chosing a transition
- I replies trying to match it
- if I succeeds, R plays again
- R wins if I fails to find a corresponding match
- I wins if it replies to all moves from R and the game is in a configuration where all states have been visited or R can't move further. In this case is said that I has a wining strategy

Examples

$$
\begin{tikzcd}
& q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
q_2 \arrow[rr, "c"] & & q_3 \arrow[loop right, "c"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
m \arrow[d, "a"'] \\
n \arrow[loop right, "c"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
q_1 \arrow[r, "a"] & q_2 \arrow[r, "a"] & q_3 \arrow[r, "a"] & \cdots
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
h \arrow[loop right, "a"]
\end{tikzcd}
$$

Bisimilarity

Definition

$$p \sim q \;\Leftrightarrow\; \langle \exists\, R \;::\; R \text{ is a bisimulation and } \langle p, q\rangle \in R \rangle$$

Lemma

1. The identity relation id is a bisimulation
2. The empty relation $\perp$ is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \mid i \in I\}$ is a bisimulation

Bisimilarity

Lemma
The bisimilarity relation is an equivalence relation
(ie, reflexive, symmetric and transitive)

Lemma
The class of all bisimulations between two LTS has the structure of a complete lattice, ordered by set inclusion, whose top is the bisimilarity relation $\sim$.

Bisimilarity

Warning

$\boxed{\text{The bisimilarity relation } \sim \text{ is not the symmetric closure of } \lesssim}$

Example

$$q_0 \lesssim p_0, \; p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$

$$\begin{tikzcd} & q_1 & \\ q_0 \arrow[ur, "a"] \arrow[dr, "a"] & & \\ & q_2 \arrow[r, "b"] & q_3 \end{tikzcd} \qquad\qquad p_0 \xrightarrow{a} p_1 \xrightarrow{b} p_3$$

Notes

Similarity as the greatest simulation

$$\lesssim \;\triangleq\; \bigcup\{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

$$\sim \;\triangleq\; \bigcup\{S \mid S \text{ is a bisimulation}\}$$

cf relational translation of definitions
$\lesssim$ and $\sim$ as greatest fix points (Tarski's theorem)