Trace equivalence and bisimilarity

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Trace

Definition

Let $T = \langle S, N, \downarrow, \longrightarrow \rangle$ be a labelled transition system. The set of traces $\mathsf{Tr}(s)$, for $s \in S$ is the minimal set satisfying

(1) $\epsilon \in \mathsf{Tr}(s)$

(2) $\checkmark \in \mathsf{Tr}(s) \;\Leftrightarrow\; s \in \downarrow$

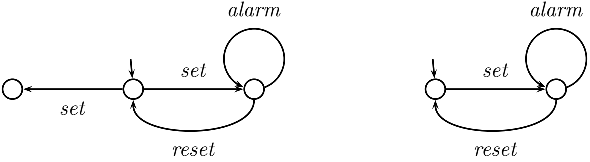
(3) $a\sigma \in \mathsf{Tr}(s) \;\Rightarrow\; \langle \exists\, s' \;:\; s' \in S :\; s \xrightarrow{a} s' \wedge \sigma \in \mathsf{Tr}(s') \rangle$

Trace equivalence

Definition

Two states s, r are trace equivalent iff $\mathsf{Tr}(s) = \mathsf{Tr}(r)$

Example



Trace equivalence applies when one can neither interact with a system, nor distinguish a slow system from one that has come to a stand still.

Simulation

the quest for a behavioural equality:
able to identify states that cannot be distinguished by any realistic form of observation

Simulation

A state q simulates another state p if every transition from q is corresponded by a transition from p and this capacity is kept along the whole life of the system to which state space q belongs to.

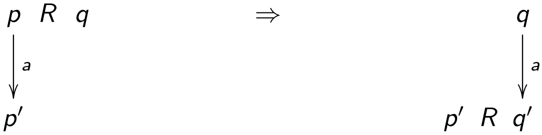
Simulation

Definition

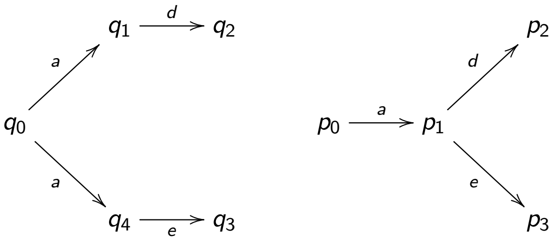
Given $\langle S_1, N, \downarrow_1, \longrightarrow_1 \rangle$ and $\langle S_2, N, \downarrow_2, \longrightarrow_2 \rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a simulation iff, for all $\langle p, q \rangle \in R$ and $a \in N$,

(1) $p \downarrow_1 \;\Rightarrow q \downarrow_2$

(2) $p \xrightarrow{a}_1 p' \;\Rightarrow\; \langle \exists\, q' \;:\; q' \in S_2 :\; q \xrightarrow{a}_2 q' \,\wedge\, \langle p', q' \rangle \in R \rangle$



Example



$q_0 \lesssim p_0$ cf. $\{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$

Similarity

Definition

$$p \lesssim q \;\equiv\; \langle \exists\, R \;::\; R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

Lemma

The similarity relation is a preorder
(ie, reflexive and transitive)

Bisimulation

Definition

Given $\langle S_1, N, \downarrow_1, \longrightarrow_1 \rangle$ and $\langle S_2, N, \downarrow_2, \longrightarrow_2 \rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a bisimulation iff both R and its converse R° are simulations.
I.e., whenever $\langle p, q \rangle \in R$ and $a \in N$,

$$(1) \quad p \downarrow_1 \ \Leftrightarrow q \downarrow_2$$

$$(2.1) \quad p \xrightarrow{a}_1 p' \ \Rightarrow \ \langle \exists\, q' \ : \ q' \in S_2 : \ q \xrightarrow{a}_2 q' \ \wedge \ \langle p', q' \rangle \in R \rangle$$

$$(2.1) \quad q \xrightarrow{a}_2 q' \ \Rightarrow \ \langle \exists\, p' \ : \ p' \in S_1 : \ p \xrightarrow{a}_1 p' \ \wedge \ \langle p', q' \rangle \in R \rangle$$

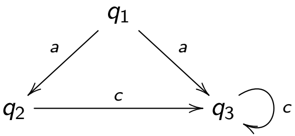
Bisimulation

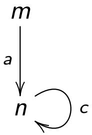
The Game characterization

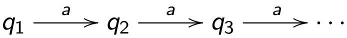
Two players R and I discuss whether the transition structures are mutually corresponding

- R starts by chosing a transition
- I replies trying to match it
- if I succeeds, R plays again
- R wins if I fails to find a corresponding match
- I wins if it replies to all moves from R and the game is in a configuration where all states have been visited or R can't move further. In this case is said that I has a wining strategy

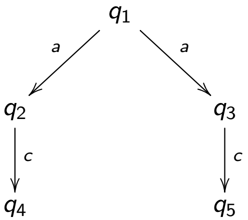
Examples

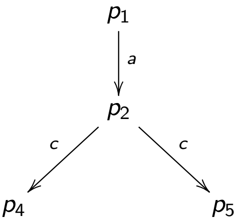






Examples


q1
a
a
q2
q3
c
c
q4
q5


p1
a
p2
c
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p4
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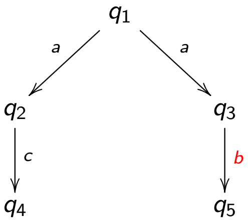
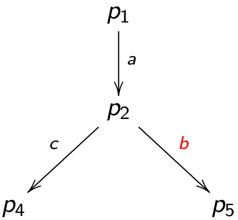

q1
a
a
q2
q3
c
b
q4
q5


p1
a
p2
c
b
p4
p5

Bisimilarity

Definition

$$p \sim q \;\equiv\; \langle \exists\, R \;::\; R \text{ is a bisimulation and } \langle p, q\rangle \in R\rangle$$

Lemma

1. The identity relation id is a bisimulation
2. The empty relation $\perp$ is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \mid i \in I\}$ is a bisimulation

Bisimilarity

Lemma

The bisimilarity relation is an equivalence relation
(ie, reflexive, symmetric and transitive)

Lemma

The class of all bisimulations between two LTS has the structure of a complete lattice, ordered by set inclusion, whose top is the bisimilarity relation $\sim$.

Bisimilarity

Lemma

In a deterministic labelled transition system, two states are bisimilar iff they are trace equivalent, i.e.,

$$s \sim s' \quad \Leftrightarrow \quad \mathrm{Tr}(s) = \mathrm{Tr}(s')$$

Hint: define a relation R as

$$\langle x, y \rangle \in R \;\Leftrightarrow\; \mathrm{Tr}(x) = \mathrm{Tr}(y)$$

and show R is a bisimulation.

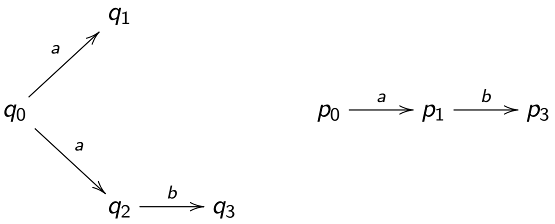
Bisimilarity

Warning

The bisimilarity relation $\sim$ **is not** the symmetric closure of $\lesssim$

Example

$$q_0 \lesssim p_0, \ p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$



Notes

Similarity as the greatest simulation

$$\lesssim \triangleq \bigcup\{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

$$\sim \triangleq \bigcup\{S \mid S \text{ is a bisimulation}\}$$