Simply-typed λ-calculus is not enough

Simply-typed λ-calculus has not enough expressive power to encode the kind of logic used in the previous example.

There are several type systems embedding some of the features described in our example. For example:

- **System F** - features polymorphism
- **λP** - features dependent types
- **System Fω**- features higher-order polymorphism
- **CC** - features dependent types and higher-order polymorphism

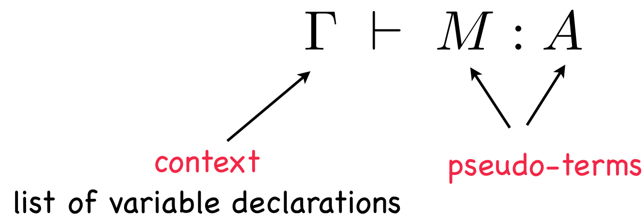
There is a general class of typed λ-calculi were all these systems can be described - the **Pure Type Systems**.

Pure Type Systems

- Pure Type Systems (PTS) provide a general description for a large class of typed λ-calculi.
- PTS make it possible to derive lot of meta theoretic properties in a generic way.
- In PTS we only have one type constructor (Π) and one computation rule (β). (Therefore the name "pure").
- PTS were originally introduced (albeit in a different from) by S. Berardi and J. Terlouw as a generalization of Barendregt's λ-cube, which itself provides a fine-grained analysis of the Calculus of Constructions.

Pure Type Systems

PTS are formal systems for deriving judgments of the form



M is of type A relative to a typing of the free variables of M and A (which are declared in Γ)

Syntax

PTS have a single category of expressions, which are called **pseudo-terms**.

The definitions of pseudo-terms is parameterized by a set $\mathcal{V}$ of **variables** and a set $\mathcal{S}$ of **sorts** (constants that denote the universes of the type system).

Definition

The set $\mathcal{T}$ of **pseudo-terms** are defined by the abstract syntax

$$\mathcal{T} \ ::= \ \mathcal{S} \mid \mathcal{V} \mid \mathcal{T}\,\mathcal{T} \mid \lambda\,\mathcal{V}{:}\mathcal{T}.\mathcal{T} \mid \Pi\,\mathcal{V}{:}\mathcal{T}.\mathcal{T}$$

Both Π and λ bind variables.
We have the usual notation for free variables and bound variables.

Definitions

Pseudo-terms inherit much of the standard definitions and notations of λ-calculi.

- FV(M) denotes the set of free variables of the pseudo-term M .
- We write $A \to B$ instead of $\Pi\, x : A.\, B$ whenever $x \notin$ FV(B).
- $M\,[x := N\,]$ denotes the substitution of N for all the free occurrences of x in M .
- We identify pseudo-terms that are equal up to a renaming of bound variables (**α-conversion**).
- We assume the standard variable convention, so all bound variables are chosen to be different from free variables.

Definitions

- **β-reduction** is defined as the compatible closure of the rule

$$(\lambda\, x : A.M)\, N \quad \to_\beta \quad M[x := N]$$

$\twoheadrightarrow_\beta$ is the reflexive-transitive closure of $\to_\beta$

$=_\beta$ is the reflexive-symmetric-transitive closure of $\to_\beta$

- Application associates to the left, abstraction to the right and application binds more tightly than abstraction.
- We let x, y, z , ... range over $\mathcal{V}$ and s, s', ... range over S

Salient Features of PTS

- PTS describe λ-calculi à la Church (λ-abstractions carry the domain of bound variables).
- PTS are minimal (just Π type construction and β reduction rule), which imposes strict limitations on their applicability.
- PTS model dependent types. Type constructor Π captures in the type theory the set-theoretic notion of generic or dependent function space.

Dependent types

In the type theory one can define for every set A and A-indexed family of sets $(B_a)_{x\in A}$ a new set $\Pi_{x\in A} B_x$ called dependent function space.

Elements of $\Pi_{x\in A} B_x$ are functions with domain A and such that $f(a) \in B_a$ for every $a \in A$.

Π-construction of PTS works in the same way:

$\Pi\, x\!:\!A.\, B(x)$ is the type of terms F such that, for every $a : A$, $F\,a : B(a)$

Specifications

The typing system of PTS is parameterized by a triple $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ where

$\mathcal{S}$ is the set of universes of the type system;
$\mathcal{A}$ determine the typing relation between universes;
$\mathcal{R}$ determine which dependent function types may be found and where they live.

Definition

A PTS-**specification** is a triple $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ where

- $\mathcal{S}$ is a set of **sorts**
- $\mathcal{A} \subseteq \mathcal{S} \times \mathcal{S}$ is a set of **axioms**
- $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{S}$ is a set of **rules**

We use $(s1,s2)$ to denote rules of the form $(s1,s2,s2)$.

Every specification $\mathbf{S}$ induces a PTS $\lambda\mathbf{S}$.

Contexts and Judgments

- The set $\mathcal{G}$ of **contexts** is given by the abstract syntax $\mathcal{G} ::= \langle\rangle \mid \mathcal{G}, \mathcal{V} : \mathcal{T}$
  - We let $\subseteq$ denote context inclusion
  - The domain of a context is defined by the clause

    $$\text{dom}(x_1 : A_1, ..., x_n : A_n) = \{x_1, ..., x_n\}$$

  - We let Γ, Δ range over $\mathcal{G}$

- A **judgment** is a triple of the form $\Gamma \vdash A : B$ where $A, B \in \mathcal{T}$ and $\Gamma \in \mathcal{G}$.

- A judgment is **derivable** if it can be inferred from the typing rules of the next slide.
  - If $\Gamma \vdash A : B$ then Γ, A and B are legal.
  - If $\Gamma \vdash A : s$ for $s \in \mathcal{S}$, we say that A is a type.

Typing rules for PTS

(axiom) $\langle\rangle \vdash s_1 : s_2$ if $(s_1, s_2) \in \mathcal{A}$

(start) $$\frac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A}$$ if $x \notin \mathsf{dom}(\Gamma)$

(weakening) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x{:}C \vdash A : B}$$ if $x \notin \mathsf{dom}(\Gamma)$

(product) $$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\Pi x{:}A.\, B) : s_3}$$ if $(s_1, s_2, s_3) \in \mathcal{R}$

(application) $$\frac{\Gamma \vdash F : (\Pi x{:}A.\, B) \quad \Gamma \vdash a : A}{\Gamma \vdash F\, a : B[x := a]}$$

(abstraction) $$\frac{\Gamma, x{:}A \vdash b : B \quad \Gamma \vdash (\Pi x{:}A.\, B) : s}{\Gamma \vdash \lambda x{:}A.b : (\Pi x{:}A.\, B)}$$

(conversion) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}$$ if $B =_\beta B'$

Typing rules for PTS

(axiom) $\langle\rangle \vdash s_1 : s_2$ if $(s_1, s_2) \in \mathcal{A}$

It embeds the relation $\mathcal{A}$ into the type system.

Typing rules for PTS

(start) $$\frac{\Gamma \vdash A : s}{\Gamma, x\!:\!A \vdash x : A}$$ if $x \notin \mathsf{dom}(\Gamma)$

(weakening) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x\!:\!C \vdash A : B}$$ if $x \notin \mathsf{dom}(\Gamma)$

It allows the introduction of variables in a context.

Typing rules for PTS

(product) $$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x\!:\!A \vdash B : s_2}{\Gamma \vdash (\Pi\, x\!:\!A.\, B) : s_3}$$ if $(s_1, s_2, s_3) \in \mathcal{R}$

It allows for dependent function types to be formed, provided they match the rule in $\mathcal{R}$.

Typing rules for PTS

$$(\text{application}) \quad \frac{\Gamma \vdash F : (\Pi\, x\!:\!A.\, B) \quad \Gamma \vdash a : A}{\Gamma \vdash F\, a : B[x := a]}$$

It allows to form applications.

Note substitution $[x := a]$ in the type of the application, in order to accommodate type dependencies.

Typing rules for PTS

$$(\text{abstraction}) \quad \frac{\Gamma, x\!:\!A \vdash b : B \quad \Gamma \vdash (\Pi\, x\!:\!A.\, B) : s}{\Gamma \vdash \lambda\, x\!:\!A.b : (\Pi\, x\!:\!A.\, B)}$$

It allows to build λ-abstractions.

Note that the side condition requires that the dependent function type is well formed.

Typing rules for PTS

$$(\text{conversion}) \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \quad \text{if } B =_\beta B'$$

It ensures that convertible types (i.e. types that are β-equal) have the same inhabitants.

This rule is crucial for higher-order type theories, because types are λ-terms and can be reduced, and for dependent type theories, because terms may occur in types.

Examples of PTS

Non-dependent type systems (i.e. an expression $M : A$ with $A : *$ cannot appear as a subexpression of $B : *$)

λ→, the simply typed λ-calculus.

$\lambda\!\to$	$\mathcal{S}$	$=$	$*,\ \square$
	$\mathcal{A}$	$=$	$(* : \square)$
	$\mathcal{R}$	$=$	$(*, *)$

λ2 is the PTS counterpart of Girard's System F.

$\lambda 2$	$\mathcal{S}$	$=$	$*,\ \square$
	$\mathcal{A}$	$=$	$(* : \square)$
	$\mathcal{R}$	$=$	$(*, *),\ (\square, *)$

λω is the PTS counterpart of Girard's System Fω.

$\lambda\omega$	$\mathcal{S}$	$=$	$*,\ \square$
	$\mathcal{A}$	$=$	$(* : \square)$
	$\mathcal{R}$	$=$	$(*, *),\ (\square, *),\ (\square, \square)$

In logical terms, these non-dependent systems correspond to propositional logics.

More examples of non-dependent PTS

λU^-, Girard's System U^-

λU^-	$\mathcal{S} = *, \Box, \triangle$ $\mathcal{A} = (* : \Box), (\Box : \triangle)$ $\mathcal{R} = (*, *), (\Box, *), (\Box, \Box), (\triangle, \Box)$

λU , System U

λU	$\mathcal{S} = *, \Box, \triangle$ $\mathcal{A} = (* : \Box), (\Box : \triangle)$ $\mathcal{R} = (*, *), (\Box, *), (\Box, \Box), (\triangle, *), (\triangle, \Box)$

The System $\lambda *$

$\lambda *$	$\mathcal{S} = *$ $\mathcal{A} = (* : *)$ $\mathcal{R} = (*, *)$

λU^-, λU and $\lambda *$ are **inconsistent** in the sense that there exists a pseudo-term M such that the judgment $A : * \vdash M : A$ is derivable.

Examples of dependent PTS

It is possible to type expressions $B : *$ which contain as subexpression $M : A : *$.

λP is the PTS counterpart of the Logical Frameworks due to Harper et al.

λP	$\mathcal{S} = *, \Box$ $\mathcal{A} = (* : \Box)$ $\mathcal{R} = (*, *), (*, \Box)$

λP2 is the PTS counterpart of Longo and Moggi's system also named λP2.

$\lambda P2$	$\mathcal{S} = *, \Box$ $\mathcal{A} = (* : \Box)$ $\mathcal{R} = (*, *), (\Box, *), (*, \Box)$

λC (also known as **λPω**) is the PTS counterpart of Coquand and Huet's Calculus of Constructions.

λC	$\mathcal{S} = *, \Box$ $\mathcal{A} = (* : \Box)$ $\mathcal{R} = (*, *), (\Box, *), (*, \Box), (\Box, \Box)$

In logical terms, these dependent systems correspond to predicate logics.