Deduction systems and intuitionism
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Program Semantics, Verification, and Construction
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Outline

(Classical) propositional logic

$\mathcal{V}_{Prop}$ infinite set of propositional variables

- $p, q, \ldots$ are formulae ($\mathcal{V}_{Prop}$);
- $\bot$ is a formula;
- if φ and ψ are formulae, then $(\varphi \rightarrow \psi)$, $(\varphi \land \psi)$, $(\varphi \lor \psi)$ and $(\neg\varphi)$ are formulae.

Semantics of (classical) propositional logic

Truth values: $\top$and $\bot$

Interpretation $v : \mathcal{V}_{Prop} \longrightarrow \{\top, \bot\}$

The interpretation can be inductively extended to the set of formulae:

φ	$\neg\varphi$
$\bot$	$\top$
$\top$	$\bot$

φ	ψ	$\varphi \land \psi$
$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$

φ	ψ	$\varphi \lor \psi$
$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\top$	$\top$	$\top$

φ	ψ	$\varphi \rightarrow \psi$
$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$

Satisfiability and Validity

A formula φ is

- satisfiable iff exists an interpretation v such that $v(\varphi) = \top$, $\models_v \varphi$ (and v satisfies φ)
  Ex: $\models p \lor q$
- valid iff for all interpretations v, $v(\varphi) = \top$, $\models \varphi$
  Ex: $\models p \lor \neg p$
- contradiction iff for all interpretations v, $v(\varphi) = \bot (\not\models \varphi)$.
  Ex: $\not\models p \land \neg p$.

Entailment

Γ a set of formulae

An interpretation v satisfies Γ iff v satisfies every formula $\psi \in \Gamma$.

Γ is satisfiable if exists an interpretation that satisfies Γ

Γ entails φ, $\Gamma \models \varphi$, iff all interpretations that satisfy Γ, satisfy also φ.

$\emptyset \models \varphi$ is equivalent to $\models \varphi$

Deduction Systems

Sets of rules and axioms from which it is possible to obtain a formula φ (considering or not an initial set of assumptions Γ): :

$$\vdash \varphi$$

or

$$\Gamma \vdash \varphi$$

If $\vdash \varphi$, φ is a theorem

The (proof) deduction systems must be sound and complete:

$$\vdash \varphi \text{ iff } \models \varphi$$

or :

$$\Gamma \vdash \varphi \text{ iff } \Gamma \models \varphi$$

Rules

A inference rule is of the form:

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

φ_i are assumptions, ψ is the conclusion

A rule without assumptions is called axiom

Deduction (derivation or proof) of φ is a tree:

- each node is labelled with a formula
- the formula of a parent node is a conclusion of a rule, which assumptions are formulae of the children nodes
- formulae of the leaves are called *initial*
- the root formula is the final formula φ

Hilbert-style system

Considering only a complete set of connectives $\{\neg, \rightarrow\}$:

Axioms

- $\varphi \rightarrow (\psi \rightarrow \varphi)$
- $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$
- $(\neg\psi \rightarrow \neg\varphi) \rightarrow ((\neg\psi \rightarrow \varphi) \rightarrow \psi)$

Inference rules

- *Modus ponens*: from φ and $\varphi \rightarrow \psi$, infer ψ

Lemma (deduction lemma)

If $\Sigma \cup \{\psi\} \vdash_H \theta$ *then* $\Sigma \vdash_H \psi \rightarrow \theta$.

Theorem

The deduction system H is sound and complete for classical propositional logic.

Natural Deduction

- System invented by G. Gentzen (1935) which rules try to reflect the usual mathematical proofs.
- It has no axioms. Only inference rules.
- For each connective, exist two types of rules:
  - introduction rules
  - elimination rules.
- The initial formulae can be assumptions introduced for the aplication of a rule:

  A sub-deduction starts and when ends, the correspondent assumptions are cancelled

Motivation for the Natural Deduction

$$(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))$$

Proof:

$$\begin{array}{c|c}
\begin{array}{c} X \\ X \vee Y \\ X \vee Z \\ ((X \vee Y) \wedge (X \vee Z)) \\ \\ \\ \end{array} &
\begin{array}{c} Y \wedge Z \\ Y \\ Z \\ X \vee Y \\ X \vee Z \\ ((X \vee Y) \wedge (X \vee Z)) \end{array}
\end{array}$$

$$\dfrac{\dfrac{[(X \vee (Y \wedge Z))] \quad \dfrac{\begin{array}{c}[X] \\ X \vee Y \\ X \vee Z\end{array}}{(X \vee Y) \wedge (X \vee Z)} \quad \dfrac{\begin{array}{c}[Y \wedge Z] \\ Y \\ Z \\ X \vee Y \\ X \vee Z\end{array}}{(X \vee Y) \wedge (X \vee Z)}}{(X \vee Y) \wedge (X \vee Z)}}{(X \vee (Y \wedge Z)) \rightarrow ((X \vee Y) \wedge (X \vee Z))}$$

Natural Deduction NK_0

	Introduction	Elimination
$\wedge$	$\dfrac{\begin{matrix}\vdots & \vdots\\ \varphi & \psi\end{matrix}}{\varphi \wedge \psi}\wedge\mathbf{I}$	$\dfrac{\begin{matrix}\vdots\\ \varphi \wedge \psi\end{matrix}}{\varphi}\wedge\mathbf{E}_1 \qquad \dfrac{\begin{matrix}\vdots\\ \varphi \wedge \psi\end{matrix}}{\psi}\wedge\mathbf{E}_2$
$\vee$	$\dfrac{\begin{matrix}\vdots\\ \varphi\end{matrix}}{\varphi \vee \psi}\vee\mathbf{I}_1 \qquad \dfrac{\begin{matrix}\vdots\\ \psi\end{matrix}}{\varphi \vee \psi}\vee\mathbf{I}_2$	$\dfrac{\begin{matrix} & [\varphi]^i & [\psi]^j\\ \vdots & \vdots & \vdots\\ \varphi \vee \psi & \gamma & \gamma\end{matrix}}{\gamma}\vee\mathbf{E},i,j$
$\rightarrow$	$\dfrac{\begin{matrix}[\varphi]^i\\ \vdots\\ \psi\end{matrix}}{\varphi\rightarrow\psi}\rightarrow\mathbf{I},i$	$\dfrac{\begin{matrix}\vdots & \vdots\\ \varphi & \varphi\rightarrow\psi\end{matrix}}{\psi}\rightarrow\mathbf{E}$

Examples

$\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$

$$\dfrac{\dfrac{[\varphi]^2}{\psi \rightarrow \varphi}\,1}{\varphi \rightarrow (\psi \rightarrow \varphi)}\,2$$

$\vdash (\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \rightarrow (\psi \rightarrow \gamma)]^1 \qquad [\varphi]^2}{\psi \rightarrow \gamma} \qquad \dfrac{[\varphi \rightarrow \psi]^3 \qquad [\varphi]^2}{\psi}}{\gamma}}{\varphi \rightarrow \gamma}\,2}{(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)}\,3}{(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)}\,1$$

Natural Deduction, NK_0 (cont.)

	Introdution	**Elimination**
$\neg$	$\dfrac{\begin{matrix}[\varphi]^i \\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\,\neg\mathbf{I},i$	$\dfrac{\begin{matrix}\vdots & & \vdots \\ \varphi & & \neg\varphi\end{matrix}}{\psi}\,\neg\mathbf{E}$
$\neg\neg$	$\dfrac{\begin{matrix}\vdots \\ \varphi\end{matrix}}{\neg\neg\varphi}\,\neg\neg\mathbf{I}$	$\dfrac{\begin{matrix}\vdots \\ \neg\neg\varphi\end{matrix}}{\varphi}\,\neg\neg\mathbf{E}$

Example

$$\vdash (\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\psi \to \neg\varphi]^1 \qquad [\neg\psi]^2}{\neg\varphi} \qquad \dfrac{[\neg\psi \to \varphi]^3 \qquad [\neg\psi]^2}{\varphi}}{\bot}}{\neg\neg\psi}\,2}{\psi}}{(\neg\psi \to \varphi) \to \psi}\,3}{(\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)}\,1$$

Modus Tollens

$$\dfrac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}\,\mathbf{MT}$$

$$\dfrac{\dfrac{\dfrac{\varphi \to \psi \qquad [\varphi]^1}{\psi} \qquad \neg\psi}{\bot}}{\neg\varphi}\,1$$

Excluded middle

$$\dfrac{}{\varphi \vee \neg\varphi}\,\mathbf{TE}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi \vee \neg\varphi)]^3 \qquad \dfrac{[\neg\varphi]^1}{\varphi \vee \neg\varphi}}{\bot}}{\neg\neg\varphi}\,1}{\varphi} \qquad \dfrac{\dfrac{[\neg(\varphi \vee \neg\varphi)]^3 \qquad \dfrac{[\varphi]^2}{\varphi \vee \neg\varphi}}{\bot}}{\neg\varphi}\,2}{\bot}}{\dfrac{\neg\neg(\varphi \vee \neg\varphi)}{\varphi \vee \neg\varphi}}\,3$$

Soundness and Completeness

Theorem

The system NK_0 is sound and complete for classic propositional logic: $\vdash^{NK_0} \varphi$ *iff* $\models \varphi$*. And* $\Gamma \vdash^{NK_0} \varphi$ *iff* $\Gamma \models \varphi$

Theorem (Decidability)

There is an algorithm which, given a finite set Γ *of formulas and a formula* φ*, decides whether* $\Gamma \vdash^{NK_0} \varphi$*.*

(Classical) First-Order Logic

Syntax

- $Var = \{x, y, \ldots, x_0, y_0, \ldots\}$ infinite set of variables
- logic connectives $\wedge$, $\vee$, $\neg$ e $\rightarrow$;
- quantifiers $\forall$ (universal) and $\exists$ (existential);
- parenthesis (and);
- a set $\mathcal{F}$ of *functional symbols* f, g, h, $\ldots$;
- a set $\mathcal{P}$ of *predicate symbols* P, Q, R, $\ldots$;
- an *arity function* $m : \mathcal{F} \cup \mathcal{R} \longrightarrow \mathbb{N}$

(Classical) First-Order Logic

Terms

The set $\mathcal{T}$ of Terms t, s, ... are inductively defined by:

- a variable is a term
- for every $f \in F$ of arity m, for all terms $t_1, \ldots, t_m$, $f(t_1, \ldots, t_m)$ is also a term

(Classical) First-Order Logic

Formulas

- if $t_1, \ldots, t_m$ are terms and $R \in \mathcal{P}$ of arity m then $R(t_1, \ldots, t_n)$ is an atomic formula.
- an atomic formula is a formula
- if φ is a formula, $\neg\varphi$ is a formula;
- if φ and ψ are formulas, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \rightarrow \psi)$ are *formulas*
- if φ is a formula and x a variable, then $\forall x\, \varphi$ and $\exists x\, \varphi$ are *formulas*.

First-Order Semantics

Interpretation

An interpretation $\mathcal{A}$ is a pair $(A, \cdot^{\mathcal{A}})$ where A is a non-empty set (domain) and $.^{\mathcal{A}}$ a function such that:

- associates to each constant c an element $c^{\mathcal{A}} \in A$
- associates to each functional symbol $f \in \mathcal{F}$ of arity $m(f) = n$, $n > 0$ a function $f^{\mathcal{A}}$ from A^n to A
- associates to each predicate symbol $R \in \mathcal{R}$ of arity $m(R) = n$, $n > 0$ a relation $R^{\mathcal{A}} \subseteq A^n$

An assignment is a function from $s : Var \longrightarrow A$. It can be naturally extended to set of terms.

First-Order Semantics

Satisfability

Given an interpretation $\mathcal{A} = (A, \cdot^{\mathcal{A}})$ and an assignment s, the satisfability relation $\mathcal{A} \models_s \phi$ is inductively defined by:

1. $\mathcal{A} \models_s t_1 = t_2$ iff $s(t_1) = s(t_2)$
2. $\mathcal{A} \models_s R(t_1, \ldots, t_n)$ iff $(s(t_1), \ldots, s(t_n)) \in R^{\mathcal{A}}$
3. $\mathcal{A} \models_s \neg\phi$ iff $\mathcal{A} \not\models_s \phi$
4. $\mathcal{A} \models_s \phi \wedge \psi$ iff $\mathcal{A} \models_s \phi$ e $\mathcal{A} \models_s \psi$
5. $\mathcal{A} \models_s \phi \vee \psi$ iff $\mathcal{A} \models_s \phi$ ou $\mathcal{A} \models_s \psi$
6. $\mathcal{A} \models_s \phi \to \psi$ iff $\mathcal{A} \not\models_s \phi$ or $\mathcal{A} \models_s \psi$
7. $\mathcal{A} \models_s \forall x\, \phi$ iff for all $a \in A$ $\mathcal{A} \models_{s[a/x]} \phi$ where:

$$s[a/x](y) = \begin{cases} s(y) & \text{se } y \neq x \\ a & \text{se } y = x \end{cases}$$

8. $\mathcal{A} \models_s \exists x\, \phi$ iff exists an $a \in A$ such that $\mathcal{A} \models_{s[a/x]} \phi$

Example

Example

Let $\mathcal{L}_N$ *be a first-order language with* $\mathcal{F}_0 = \{0, 1\}$, $\mathcal{F}_2 = \{+, \times\}$ *e* $\mathcal{R}_2 = \{<\}$
Let $\mathcal{N} = (\mathbb{N}, \cdot^{\mathcal{N}})$ *be an interpretation with* $\cdot^{\mathcal{N}}$ *given by:*

- $0^{\mathcal{N}} = 0$, $1^{\mathcal{N}} = 1$
- $+^{\mathcal{N}}(n, m) = n + m$, $\times^{\mathcal{N}}(n, m) = n \times m$
- $<^{\mathcal{N}} = \{(n, m) \in \mathbb{N}^2 \mid n < m\}$

For every assignment $s : Var \longrightarrow \mathbb{N}$*:*
$\mathcal{N} \models_s \forall x \ < (x, +(x, 1))$

Satisfability, validity and entailment

A formula φ is

satisfiable iff exists an interpretation $\mathcal{A}$ and an assignment s, such that $\mathcal{A} \models_s \varphi$ (A is a model of ϕ)

valid iff for all interpretation $\mathcal{A}$ and assignment s, $\mathcal{A} \models_s \varphi$ ($\models \varphi$)

A **model** of a set of formulas Γ is a model of all formulas of Γ, $\mathcal{A} \models_s \Gamma$.

φ **entails** Γ, $\Gamma \models \varphi$, if every model of Γ is also a model of φ.

Natural Deduction, *NK*: quantifiers rules

	Introduction	**Elimination**
$\forall$	$\dfrac{\begin{matrix}[v] \\ \vdots \\ \varphi[v/x]\end{matrix}}{\forall x\,\varphi}\,\forall\mathbf{I}\quad (a)$	$\dfrac{\forall x\,\varphi}{\varphi[t/x]}\,\forall\mathbf{E}\quad (b)$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\,\varphi}\,\exists\mathbf{I}\quad (b)$	$\dfrac{\exists x\,\varphi \qquad \begin{matrix}[v \quad \varphi[v/x]] \\ \vdots \\ \psi\end{matrix}}{\psi}\,\exists\mathbf{E}\quad (c)$

(a) where v is a new variable

(b) where x is free for t in φ

(c) where v is a new variable and not in ψ

Example

$\neg\forall x\neg\varphi \rightarrow \exists x\varphi$

$$\dfrac{\dfrac{\dfrac{[\neg\forall x\neg\varphi]^3 \qquad \dfrac{\dfrac{\dfrac{[\neg\exists x\varphi]^2 \qquad \dfrac{[\varphi[u/x]]^1}{\exists x\varphi}}{\bot}}{\neg\varphi[u/x]}\,1}{\forall x\neg\varphi}}{\bot}}{\neg\neg\exists x\varphi}\,2}{\dfrac{\exists x\varphi}{\neg\forall x\neg\varphi \rightarrow \exists x\varphi}\,3}$$

Soundness and Completeness

Theorem

The system NK is sound and complete for classical first-order logic: $\vdash^{NK} \varphi$ *iff* $\models \varphi$*. And* $\Gamma \vdash^{NK} \varphi$ *iff* $\Gamma \models \varphi$

Theorem (Undecidability)

The set of valid formulas of a first-order logic is recursivelly enumerable but not recursive, i.e., it is undecidable to determine if a formula is a theorem.

Sequents

In each step of a tree derivation it is not easy to know which are the open assumptions:

If φ depends on open assumptions $\varphi_1, \ldots, \varphi_k$:

$$\varphi_1 \wedge \ldots \wedge \varphi_k \rightarrow \varphi$$

Sequents

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$$

Meaning: $\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \psi_1 \vee \ldots \vee \psi_m$

Empty antecedent: $\psi_1 \vee \ldots \vee \psi_m$

Empty consequent: $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$

Both empty: $\perp$

NK_0 in *sequent form*

Γ (context) is a set of formulas

$$\dfrac{}{\Gamma,\varphi \Rightarrow \varphi}\,\textbf{Ax}$$

	Introdution	**Elimination**
$\wedge$	$\dfrac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\wedge\textbf{I}$	$\dfrac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi}\wedge\textbf{E}_1$ $\quad$ $\dfrac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}\wedge\textbf{E}_2$
$\vee$	$\dfrac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\vee\textbf{I}_1$ $\quad$ $\dfrac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\vee\textbf{I}_2$	$\dfrac{\Gamma \Rightarrow \varphi \vee \psi \quad \Gamma,\varphi \Rightarrow \gamma \quad \Gamma,\psi \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\vee\textbf{E}$
$\rightarrow$	$\dfrac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi\rightarrow\psi}\rightarrow\textbf{I}$	$\dfrac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \varphi\rightarrow\psi}{\Gamma \Rightarrow \psi}\rightarrow\textbf{E}$
$\neg$	$\dfrac{\Gamma,\varphi \Rightarrow \bot}{\Gamma \Rightarrow \neg\varphi}\neg\textbf{I}$	$\dfrac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \neg\varphi}{\Gamma \Rightarrow \psi}\neg\textbf{E}$
$\neg\neg$	$\dfrac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \neg\neg\varphi}\neg\neg\textbf{I}$	$\dfrac{\Gamma \Rightarrow \neg\neg\varphi}{\Gamma \Rightarrow \varphi}\neg\neg\textbf{E}$

Deductions with *sequents*

Nodes of the derivation trees are *sequents* and $\vdash \Gamma \Rightarrow \varphi$ means $\Gamma \vdash \varphi$

$\vdash \Rightarrow \varphi \rightarrow (\psi \rightarrow \varphi)$

$$\dfrac{\dfrac{\varphi,\psi \Rightarrow \varphi}{\varphi \Rightarrow \psi\rightarrow\varphi}(\rightarrow\textbf{I})}{\Rightarrow \varphi\rightarrow(\psi\rightarrow\varphi)}(\rightarrow\textbf{I})$$

Weakening

If $\Gamma \Rightarrow \varphi$ is derivable, then for all $\Gamma' \supseteq \Gamma$, $\Gamma' \Rightarrow \varphi$ is derivable

NK in *sequent form*: quantifier rules

$$\frac{\Gamma \;\Rightarrow\; \varphi[v/x]}{\Gamma \;\Rightarrow\; \forall x\, \varphi}\,\forall\mathbf{I} \quad (a)$$

$$\frac{\Gamma \;\Rightarrow\; \forall x\, \varphi}{\Gamma \;\Rightarrow\; \varphi[t/x]}\,\forall\mathbf{E} \quad (b)$$

$$\frac{\Gamma \;\Rightarrow\; \varphi[t/x]}{\Gamma \;\Rightarrow\; \exists x\, \varphi}\,\exists\mathbf{I} \quad (b)$$

$$\frac{\Gamma \;\Rightarrow\; \exists x\, \varphi \qquad \Gamma, \varphi[v/x] \;\Rightarrow\; \psi}{\Gamma \;\Rightarrow\; \psi}\,\exists\mathbf{E} \quad (c)$$

(a) where v is a new variable, not in Γ

(b) where x is free for t in φ

(c) where v is a new variable, not in Γ and not in ψ

Gentzen *Sequent* calculi

Deductive systems introduced by Gentzen (1935) in order to obtain deduction normal forms.

Allows decidability algorithms without using completeness (semantics).

Modus ponens:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

given ψ, φ can be any formula...

Although normalisation can be obtained for NK_0, these systems are more structured and reveal the structure of the deductions:are the base of other *analytic deduction systems* as resolution and *tableaux*.

Two types of introduction rules: in the antecedent (**L**) and in the consequent (**R**).

Sequent calculus LK_0

$$\frac{}{\Gamma,\varphi \Rightarrow \Delta,\varphi}\,\textbf{Ax}$$

$$\frac{\Gamma \Rightarrow \Delta,\varphi \qquad \Gamma',\varphi \Rightarrow \Delta'}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta'}\,\textbf{Cut}$$

$$\frac{\Gamma,\varphi,\psi \Rightarrow \Delta}{\Gamma,\varphi \wedge \psi \Rightarrow \Delta}\,\wedge\textbf{L} \qquad \frac{\Gamma \Rightarrow \Delta,\varphi \qquad \Gamma \Rightarrow \Delta,\psi}{\Gamma \Rightarrow \Delta,\varphi \wedge \psi}\,\wedge\textbf{R}$$

$$\frac{\Gamma,\varphi \Rightarrow \Delta \qquad \Gamma,\psi \Rightarrow \Delta}{\Gamma,\varphi \vee \psi \Rightarrow \Delta}\,\vee\textbf{L} \qquad \frac{\Gamma \Rightarrow \Delta,\varphi,\psi}{\Gamma \Rightarrow \Delta,\varphi \vee \psi}\,\vee\textbf{R}$$

$$\frac{\Gamma \Rightarrow \varphi,\Delta \qquad \Gamma,\psi \Rightarrow \Delta}{\Gamma,\varphi\to\psi \Rightarrow \Delta}\,\to\textbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \Delta,\psi}{\Gamma \Rightarrow \Delta,\varphi\to\psi}\,\to\textbf{R}$$

$$\frac{\Gamma \Rightarrow \Delta,\varphi}{\Gamma,\neg\varphi \Rightarrow \Delta}\,\neg\textbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\neg\varphi}\,\neg\textbf{R}$$

$$\vdash \Rightarrow ((p \to q) \to p) \to p$$

$$\frac{\dfrac{\dfrac{p \Rightarrow p,q}{\Rightarrow p,(p\to q)}(\to\textbf{R}) \qquad p \Rightarrow p}{(p\to q)\to p) \Rightarrow p}(\to\textbf{L})}{\Rightarrow ((p\to q)\to p)\to p}(\to\textbf{R})$$

$$\vdash \Rightarrow (p \to (p \to q)) \to (p \to q)$$

$$\frac{\dfrac{\dfrac{p \Rightarrow p,q \qquad \dfrac{p \Rightarrow p,q \quad p,q \Rightarrow q}{(p\to q),p \Rightarrow q}(\to\textbf{L})}{p\to(p\to q),p \Rightarrow q}(\to\textbf{L})}{(p\to(p\to q) \Rightarrow (p\to q)}(\to\textbf{R})}{\Rightarrow (p\to(p\to q))\to(p\to q)}(\to\textbf{R})$$

NK_0 versus LK_0

Rules

NK_0 (in *sequents*)	LK_0
Axiom	Axiom
Introduction($\circ$**I**)	Introduction in the consequent ($\circ$**R**)
Elimination ($\circ$**E**)	Introduction in the antecedent ($\circ$**L**)

Subformula property

In a cut-free deduction of $\Gamma \Rightarrow \Delta$, all sequents are composed of subformulas of formulas in Γ, Δ only

Then is possible to obtain an algorithm that searches a proof from the root

Cut elimination

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma', \varphi \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\textbf{Cut}$$

φ cut formula

Theorem

(Hauptsatz) The deduction system LK_0, without cut rule, is sound and complete.
There is an algorithm that takes a deduction in LK_0, and turns it into a cut-free deduction of the same sequent.

Why this rule?:

- allows shorter deductions
- isolates the redundancy of the deductions
- is easier to obtain certain theoretical results
- there are systems that recover part of its functionality...preserving a normal form (*Tableaux* KE)

Idea of the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma,\varphi \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \varphi\to\psi}(\to\mathbf{R}) \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma,\psi \Rightarrow \gamma\end{array}}{\Gamma,\varphi\to\psi \Rightarrow \gamma}(\to\mathbf{L})}{\Gamma \Rightarrow \gamma}\,\mathbf{Cut}$$

Transformation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma,\varphi \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \psi}\,\mathbf{Cut} \qquad \begin{array}{c}\vdots\\ \Gamma,\psi \Rightarrow \gamma\end{array}}{\Gamma \Rightarrow \gamma}\,\mathbf{Cut}$$

- Transform applications of the cut rule in others with simple cut formulas
- Cut rules applications in upper nodes of the derivation tree
- Requires double induction: in the depth of the cut rule applications and in the complexity of the cut formula.

LK calculus: quantifier rules

$$\dfrac{\Gamma \Rightarrow \varphi[v/x], \Delta}{\Gamma \Rightarrow \forall x\, \varphi, \Delta}\ \forall\mathbf{R} \quad \text{(a)}$$

$$\dfrac{\Gamma, \varphi[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, \varphi \Rightarrow \Delta}\ \forall\mathbf{L} \quad \text{(b)}$$

$$\dfrac{\Gamma \Rightarrow \varphi[t/x], \Delta}{\Gamma \Rightarrow \exists x\, \varphi, \Delta}\ \exists\mathbf{R} \quad \text{(b)}$$

$$\dfrac{\Gamma, \varphi[v/x] \Rightarrow \Delta}{\Gamma, \exists x\, \varphi \Rightarrow \Delta}\ \exists\mathbf{L} \quad \text{(a)}$$

(a) where v is a new variable, not in Γ, Δ

(b) where x is free for t in φ

Methods of proof

The semantics of classical logic is based on the notion of *truth*. Each proposition is absolutely true or false.

Principle of the excluded middle: $p \vee \neg p$.

But that gives not much information...

> *There are two irrational numbers b and c, such that b^c is rational*

Proof by case analysis:

if $\sqrt{2}^{\sqrt{2}}$ is a rational number then we can take $b = c = \sqrt{2}$; otherwise take $b = \sqrt{2}^{\sqrt{2}}$ and $c = \sqrt{2}$.

But which are these values?...The problem is that the proof is not *constructive*

Intuitionism

- φ is true if we can prove it.
- φ is false if we can show that if we have a proof of φ we get a contradiction.

Proofs can be interpreted by sets and functions

Informal constructive semantics of connective (*BHK*-interpretation)

- A construction of $\varphi \wedge \psi$ consists of a construction of φ and a construction of ψ ($\{(a, b) \mid a \in \varphi, b \in \psi\}$)
- A construction of $\varphi \vee \psi$ consists of a construction of φ or a construction of ψ ($\{(0, a) \mid a \in \varphi\} \cup \{(1, b) \mid b \in \psi\}$)
- A construction of $\varphi \rightarrow \psi$ is a method (function) transforming every construction of φ into a construction of ψ ($\{f \mid \forall a \in \varphi, f(a) \in \psi\}$)
- A construction of $\neg\varphi$ is a method transforming every construction of φ into a no existent object
- There is no construction of $\perp$ ($\emptyset$)

Intuitionism

Show that

$p \to \neg\neg p$ (or $p \to ((p \to \bot) \to \bot)$) is intuitionistically valid:

Given a proof of p, we can obtain a proof of $((p \to \bot) \to \bot)$: Take a proof of $(p \to \bot)$. It is a method to transform proofs of p into proofs of $\bot$. Since we have a proof of p, we can obtain a proof of $\bot$.

But

$\neg\neg p \to p$ is not intuicionistically valid: the fact of not having a proof of $\neg p$, does not allow to conclude that we have a proof of p...

In the same way, $p \vee \neg p$ is not valid!... in general, is nor possible to ensure that we have a proof of p or of $\neg p$.

Intuicionistic logic is obtained from classic logic by eliminating the rule $\neg\neg$**E** in the NK_0 system...

Structural rules for *sequents*

Allow the explicit manipulation of assumptions in a proof.

In a *sequent* $\Gamma \Rightarrow \Delta$, Γ and Δ are lists of formulas.

Weakening

$$\frac{\Gamma \Rightarrow \Delta}{\varphi,\Gamma \Rightarrow \Delta} WL \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\varphi} WR$$

Contraction

$$\frac{\varphi,\varphi,\Gamma \Rightarrow \Delta}{\varphi,\Gamma \Rightarrow \Delta} CL \qquad \frac{\Gamma \Rightarrow \Delta,\varphi,\varphi}{\Gamma \Rightarrow \Delta,\varphi} CR$$

Permutation

$$\frac{\Lambda,\varphi,\psi,\Gamma \Rightarrow \Delta}{\Lambda,\psi,\varphi,\Gamma \Rightarrow \Delta} XL \qquad \frac{\Gamma \Rightarrow \Delta,\varphi,\psi,\Lambda}{\Gamma \Rightarrow \Delta,\psi,\varphi,\Lambda} XR$$

Intuitionistic Natural Deduction NJ_0

In intuitionistic logic, in a sequent $\Gamma \Rightarrow \Delta$, Δ has at most one formula.

$$\frac{}{\varphi \Rightarrow \varphi}\textbf{Ax}$$

$$\frac{\Gamma \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}W \quad \frac{\Gamma,\varphi,\varphi \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}C \quad \frac{\Gamma,\varphi,\psi,\Delta \Rightarrow \varphi'}{\Gamma,\psi,\varphi,\Delta \Rightarrow \varphi'}X$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \varphi \wedge \psi}\wedge\textbf{I} \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi}\wedge\textbf{E}_1 \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}\wedge\textbf{E}_2$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\vee\textbf{I}_1 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\vee\textbf{I}_2 \qquad \frac{\Gamma \Rightarrow \varphi \vee \psi \qquad \Delta,\varphi \Rightarrow \gamma \qquad \Delta,\psi \Rightarrow \gamma}{\Gamma,\Delta \Rightarrow \gamma}\vee\textbf{E}$$

$$\frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\to\textbf{I} \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \varphi \to \psi}{\Gamma,\Delta \Rightarrow \psi}\to\textbf{E}$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\bot\textbf{E}$$

Intuitionistic Natural Deduction NJ_0

Negation can be defined by $\neg\varphi = \varphi \to \bot$

$\vdash_{NJ_0} \Rightarrow p \to \neg\neg p$

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p \qquad p \to \bot \Rightarrow p \to \bot}{p, p \to \bot \Rightarrow \bot}}{p \Rightarrow (p \to \bot) \to \bot}}{\Rightarrow p \to ((p \to \bot) \to \bot)}$$

$\not\vdash_{NJ_0} \Rightarrow \neg\neg p \to p$...

de Morgan laws are not valid any more...

$\Rightarrow p \vee \neg p$ is not derivable

The last rule should be $\vee$ **I**. Then $\Rightarrow p$ or $\Rightarrow \neg p$ are derivable. $\Rightarrow p$ is not derivable. For $\Rightarrow \neg p$, i.e., $\Rightarrow p \to \bot$, the last rule must be $\to$ **I**, but no rule has a conclusion $p \Rightarrow \bot$.

Intuitionistic *Sequent* Calculus, LJ_0

$$\frac{}{\varphi \Rightarrow \varphi}\mathbf{Ax}$$

$$\frac{\Gamma \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}W \quad \frac{\Gamma,\varphi,\varphi \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}C \quad \frac{\Gamma,\varphi,\psi,\Delta \Rightarrow \varphi'}{\Gamma,\psi,\varphi,\Delta \Rightarrow \varphi'}X \quad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\bot$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta,\varphi \Rightarrow \varphi'}{\Gamma,\Delta \Rightarrow \varphi'}\mathbf{Cut}$$

$$\frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma,\varphi \wedge \varphi' \Rightarrow \psi} \wedge \mathbf{L}_1 \quad \frac{\Gamma,\varphi' \Rightarrow \psi}{\Gamma,\varphi \wedge \varphi' \Rightarrow \psi} \wedge \mathbf{L}_2 \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \varphi \wedge \psi} \wedge \mathbf{R}$$

$$\frac{\Gamma,\varphi \Rightarrow \psi \qquad \Delta,\varphi' \Rightarrow \psi}{\Gamma,\Delta,\varphi \vee \varphi' \Rightarrow \psi} \vee \mathbf{L} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \vee \mathbf{R}_2 \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \vee \mathbf{R}_1$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta,\psi \Rightarrow \varphi'}{\Gamma,\Delta,\varphi\to\psi \Rightarrow \varphi'}\to\mathbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi\to\psi}\to\mathbf{R}$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma,\neg\varphi \Rightarrow \bot}\neg\mathbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \bot}{\Gamma \Rightarrow \neg\varphi}\neg\mathbf{R}$$

Intuitionistic *Sequent* Calculus, LJ_0

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\neg p \Rightarrow \neg p}(Ax)}{\neg p, \neg\neg p \Rightarrow \bot}(\neg L)}{\neg p \Rightarrow \neg\neg\neg p}(\neg R)}{\Rightarrow \neg p \to \neg\neg\neg p}(\to R)$$

Proposition

$\Gamma \vdash_{LJ_0} \varphi$ *iff* $\Gamma \vdash_{NJ_0} \varphi$

Idea of the proof

($\Rightarrow$)

In the mapping $\mathcal{N}$ from proofs of LJ_0 to proofs in NJ_0, a natural deduction NJ_0 version of the cut rule is used:

$$\frac{\Gamma \;\Rightarrow\; \varphi \qquad \Delta, \varphi \;\Rightarrow\; \varphi'}{\Gamma, \Delta \;\Rightarrow\; \varphi'} Subs$$

Which is a derivable extension:

$$\frac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \varphi\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \Delta, \varphi \Rightarrow \varphi'\end{matrix}}{\Delta \Rightarrow \varphi \to \varphi'} \to I}{\Gamma, \Delta \;\Rightarrow\; \varphi'} \to E$$

Idea of the proof

($\wedge L_1$)

$$\frac{\dfrac{\pi}{\Gamma, \varphi \Rightarrow \varphi'}}{\Gamma, \varphi \wedge \psi \Rightarrow \varphi'}$$

is mapped to:

$$\frac{\dfrac{\overline{\varphi \wedge \psi \Rightarrow \varphi \wedge \psi}\,(Ax)}{\varphi \wedge \psi \Rightarrow \varphi} \wedge E \qquad \dfrac{N(\pi)}{\Gamma, \varphi \Rightarrow \varphi'}}{\Gamma, \varphi \wedge \psi \Rightarrow \varphi'} Subs$$

($\to L$)

$$\frac{\dfrac{\pi}{\Gamma \Rightarrow \varphi} \qquad \dfrac{\pi'}{\Delta, \psi \Rightarrow \varphi'}}{\Gamma, \Delta, \varphi \to \psi \Rightarrow \varphi'}$$

is mapped to :

$$\frac{\dfrac{\overline{\varphi \to \psi \Rightarrow \varphi \to \psi}\,(Ax) \quad \dfrac{N(\pi)}{\Gamma \Rightarrow \varphi}}{\Gamma, \varphi \to \psi \Rightarrow \psi} \to E \qquad \dfrac{N(\pi')}{\Delta, \psi \Rightarrow \varphi'}}{\Gamma, \Delta, \varphi \to \psi \Rightarrow \varphi'} Subs$$

etc...

Idea of the proof

- Cut-free deductions are transformed with the *Subs* rule
- For the right rules the cut rule is used...

...So this transformation do not preserve normal forms...

...But natural deductions can be normalized by elimination of consecutive applications of an introduction rule and an elimination rule for the same connective (this is also called a **cut**).

Applying these process the effect of the **Subs** rule can be eliminated...

Normalization in NJ_0 (without *sequents*...)

$$\dfrac{\dfrac{\dfrac{[\varphi]^{(1)}}{\varphi\to\varphi}{(1)} \qquad \dfrac{[\psi]^{(2)}}{\psi\to\psi}{(2)}}{(\varphi\to\varphi)\wedge(\psi\to\psi)}}{\varphi\to\varphi}$$

Simplifies to:

$$\dfrac{[\varphi]^{(1)}}{\varphi\to\varphi}{(1)}$$

$$\dfrac{\dfrac{[\varphi]^{(3)}}{\varphi\to\varphi}{(3)} \qquad \dfrac{\dfrac{\dfrac{[\varphi\to\varphi]^{(1)}}{\psi\to\varphi\to\varphi}{(2)}}{(\varphi\to\varphi)\to\psi\to\varphi\to\varphi}}{}}{\psi\to\varphi\to\varphi}$$

Simplifies to:

$$\dfrac{\dfrac{[\varphi]^{(3)}}{\varphi\to\varphi}{(3)}}{\psi\to\varphi\to\varphi}{(2)}$$

Normalization rules for ($\rightarrow$ I e $\wedge$ I)

$$\dfrac{\dfrac{\Sigma}{\psi} \qquad \dfrac{\dfrac{\dfrac{[\psi]^{(i)}}{\Pi}}{\varphi}}{\psi \rightarrow \varphi}(i)}{\varphi} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\Sigma}{\psi}}{\Pi}}{\varphi}$$

$$\dfrac{\dfrac{\dfrac{\Sigma}{\psi} \qquad \dfrac{\Pi}{\varphi}}{\psi \wedge \varphi}}{\varphi} \quad \Rightarrow \quad \dfrac{\Sigma}{\varphi}$$

In the next session, compare with normalization in the λ-calculus...

Semantics of intuitionistic logic

The deduction systems NJ_0 and LJ_0 are sound for classic truth valued semantics but they are not complete...

[Gödel] There is no finite truth table semantics for intuitionistic logic that is sound and complete

Boolean algebras can be modified for intuitionistic logic (Heyting algebras), by introducing the notion of partiality.

The same can be done with notion of *possible worlds*.

Kripke semantics

A Kripke frame $\mathcal{F}$

is a structure $(X, \leq, \models)$ where:

- $(X, \leq)$ is a partial order
- $\models$ is a binary relation in $X \times \mathcal{V}_{Prop}$ such that for all $x, y \in X$, if $x \models p$ and $x \leq y$ then $y \models p$.

Semantics

If $x \models p$, p is forced at x.
$\models$ extends to the set of formulae:
$x \models \varphi \wedge \psi$ iff $x \models \varphi$ and $x \models \psi$
$x \models \varphi \vee \psi$ iff $x \models \varphi$ or $x \models \psi$
$x \models \varphi \rightarrow \psi$ iff $\forall y$, $x \leq y$, $y \models \varphi$ then $y \models \psi$
$x \models \neg\varphi$ iff $\forall y$, $x \leq y$, $y \not\models \varphi$
$x \not\models \bot$, $\forall x$

Kripke semantics

$x \models \Gamma \Rightarrow \psi$ iff $(\forall \varphi \in \Gamma, x \models \varphi)$ then $x \models \psi$.

A formula φ is forced in $\mathcal{F}$ if every $x \in X$ forces φ

φ is intuitionistically valid if it is forced in every frame $\mathcal{F}$

Theorem (Monotonicity)

If $x \models \varphi$ and $x \leq y$ then $y \models \varphi$

Proposition

$x \models \varphi \rightarrow \bot$ *iff* $x \models \neg\varphi$

Proposition

$x \models \neg\neg\varphi$ *iff* $\forall y$, $x \leq y$, $\exists u$, $y \leq u$ *such that* $u \models \varphi$

Examples

- $\varphi \rightarrow \varphi$ is intuitionistically valid
  $x \models \varphi \rightarrow \varphi$ iff $\forall y,\ x \leq y,\ y \models \varphi$ then $y \models \varphi$
- $\varphi \rightarrow \neg\neg\varphi$ is intuitionistically valid
  $x \models \varphi \rightarrow \neg\neg\varphi$ iff $\forall y,\ x \leq y,\ y \models \varphi$ then $y \models \neg\neg\varphi$. And $y \models \neg\neg\varphi$ if $\forall u,\ y \leq u, \exists v,\ u \leq v$ and $v \models \varphi$. The result follows by transitivity of $\leq$ and monotonicity.
- $\neg\neg\varphi \rightarrow \varphi$ is not intuitionistically valid
  $x \models \neg\neg\varphi \rightarrow \varphi$, iff $\forall y,\ x \leq y,\ y \models \neg\neg\varphi$ then $y \models \varphi$. I.e., $\forall u,\ y \leq u$, $\exists v,\ u \leq v,\ v \models \varphi$ implies that $y \models \varphi$. For instance, take $(\{0, 00\}, (0 \leq 00), \models)$ with $00 \models p$.
- $p \ \vee \ \neg p$ is not intuitionistically valid (use the same frame as above).

Kripke semantics

Theorem (Soundness)

If $\Gamma \Rightarrow \varphi$ is derivable in NJ_0 then $\Gamma \Rightarrow \varphi$ is intuitionistically valid.

Theorem (Completeness)

If $\Gamma \Rightarrow \varphi$ is intuitionistically valid, then $\vdash_{NJ_0} \Gamma \Rightarrow \varphi$.

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