Pure λ-calculus

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- concept of function seen as a rule, i.e. process of passing an argument to a value (contrary to the notion of seeing a function as a graph);
- this is important for the study of computability and for theory of computation in general, since it emphasizes the computational aspect associated to the notion of function.

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Examples: $(\lambda x.x)$, $(x(\lambda y.(xy)))$,...

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Examples: $\lambda xy.xyz \equiv_\alpha \lambda yu.yuz$, but $(\lambda x.x)z \not\equiv_\alpha (\lambda x.y)z$

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- the term $(\lambda xy.x)(\lambda x.x)((\lambda x.xx)(\lambda x.xx))$ has normal form $\lambda x.x$, but not every reduction sequence leads to this normal form.

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Theorem: (Church-Rosser) If $M \twoheadrightarrow_\beta N_1$ and $M \twoheadrightarrow_\beta N_2$, then there is a term P such that $N_1 \twoheadrightarrow_\beta P$ and $N_2 \twoheadrightarrow_\beta P$.

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Structure of β-nfs: Every β-normal form M is of the form

$$\lambda x_1 \ldots x_n.yN_1 \ldots N_m$$

with $n, m \geq 0$ and such that $N_1, \ldots, N_m$ are terms in β-normal form.

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- every term has exactly one η-nf;
- the *η-family* of a term M is the (finite) set of all terms N such that $M \rightarrow_{\eta} N$.

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- if M is a β-nf, then all members of its η-family are β-nfs and exactly one of them is a $\beta\eta$-nf.

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- $\texttt{cons} = \lambda xy.\texttt{pair}\,\texttt{false}\,(\texttt{pair}\,xy)$;
- $\texttt{null} = \texttt{fst}$;
- $\texttt{hd} = \lambda z.\texttt{fst}\,(\texttt{snd}\,z)$;
- $\texttt{tl} = \lambda z.\texttt{snd}\,(\texttt{snd}\,z)$.

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- define the functions `fact` and `tail`.

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- M is a BCI*-term* iff for every subterm of the form $\lambda x.N$ of M, x occurs *exactly once* free in N.