Deduction systems and intuitionism
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Program Semantics, Verification, and Construction
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(Classical) propositional logic

$\mathcal{V}_{Prop}$ infinite set of propositional variables

- $p, q, \ldots$ are formulae ($\mathcal{V}_{Prop}$);
- $\perp$ is a formula;
- if φ and ψ are formulae, then $(\varphi \to \psi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\neg\varphi)$ are formulae.

Semantics of (classical) propositional logic

Truth values: $\top$and $\bot$

Interpretation $v : \mathcal{V}_{Prop} \longrightarrow \{\top, \bot\}$

The interpretation can be inductively extended to the set of formulae:

φ	$\neg\ \varphi$
$\bot$	$\top$
$\top$	$\bot$

φ	ψ	$\varphi \wedge\ \psi$
$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$

φ	ψ	$\varphi \vee \psi$
$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\top$	$\top$	$\top$

φ	ψ	$\varphi \rightarrow \psi$
$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$

Satisfiability and Validity

A formula φ is

satisfiable iff exists an interpretation v such that $v(\varphi) = \top$, $\models_v \varphi$ (and v satisfies φ)
Ex: $\models p \ \vee \ q$

valid iff for all interpretations v, $v(\varphi) = \top$, $\models \varphi$
Ex: $\models p \ \vee \ \neg p$

contradiction iff for all interpretations v, $v(\varphi) = \bot (\not\models \varphi)$.
Ex: $\not\models p \ \wedge \ \neg p$.

Entailment

Γ a set of formulae

An interpretation v satisfies Γ iff v satisfies every formula $\psi \in \Gamma$.

Γ is satisfiable if exists an interpretation that satisfies Γ

Γ entails φ, $\Gamma \models \varphi$, iff all interpretations that satisfy Γ, satisfy also φ.

$\emptyset \models \varphi$ is equivalent to $\models \varphi$

Deduction Systems

Sets of rules and axioms from which it is possible to obtain a formula φ (considering or not an initial set of assumptions Γ): :

$$\vdash \varphi$$

or

$$\Gamma \vdash \varphi$$

If $\vdash \varphi$, φ is a theorem

The (proof) deduction systems must be sound and complete:

$$\vdash \varphi \text{ iff } \models \varphi$$

or :

$$\Gamma \vdash \varphi \text{ iff } \Gamma \models \varphi$$

Rules

A inference rule is of the form:

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

φ_i are assumptions, ψ is the conclusion

Deduction (derivation or proof) of φ is a tree:

- each node is labelled with a formula
- the formula of a parent node is a conclusion of a rule, which assumptions are formulae of the children nodes
- formulae of the leaves are called *initial*
- the root formula is the final formula φ

Hilbert-style system

Considering only a complete set of connectives $\{\neg, \rightarrow\}$:

Axioms

- $\varphi \rightarrow (\psi \rightarrow \varphi)$
- $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$
- $(\neg\psi \rightarrow \neg\varphi) \rightarrow ((\neg\psi \rightarrow \varphi) \rightarrow \psi)$

Inference rules

- *Modus ponens*: from φ and $\varphi \rightarrow \psi$, infer ψ

Lemma (deduction lemma)

If $\Sigma \cup \{\psi\} \vdash_H \theta$ *then* $\Sigma \vdash_H \psi \rightarrow \theta$.

Theorem

The deduction system H is sound and complete for classical propositional logic.

Natural Deduction

- System invented by G. Gentzen (1935) which rules try to reflect the usual mathematical proofs.
- It has no axioms. Only inference rules.
- For each connective, exist two types of rules:
  - introduction rules
  - elimination rules.
- The initial formulae can be assumptions introduced for the aplication of a rule:

A sub-deduction starts and when ends, the correspondent assumptions are cancelled

Motivation for the Natural Deduction

$$(X \lor (Y \land Z)) \to ((X \lor Y) \land (X \lor Z))$$

Proof:

X	$Y \land Z$
$X \lor Y$	Y
$X \lor Z$	Z
$((X \lor Y) \land (X \lor Z))$	$X \lor Y$
	$X \lor Z$
	$((X \lor Y) \land (X \lor Z))$

$$\dfrac{[(X \vee (Y \wedge Z))] \qquad \dfrac{\begin{array}{c}[X]\\ X \vee Y\\ X \vee Z\end{array}}{(X \vee Y) \wedge (X \vee Z)} \qquad \dfrac{\begin{array}{c}[Y \wedge Z]\\ Y\\ Z\\ X \vee Y\\ X \vee Z\end{array}}{(X \vee Y) \wedge (X \vee Z)}}{\dfrac{(X \vee Y) \wedge (X \vee Z)}{(X \vee (Y \wedge Z)) \to ((X \vee Y) \wedge (X \vee Z))}}$$

Natural Deduction NK_0

	Introduction		**Elimination**	
$\wedge$	$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \varphi & \psi\end{array}}{\varphi \wedge \psi} \wedge \mathbf{I}$		$\dfrac{\begin{array}{c}\vdots\\ \varphi \wedge \psi\end{array}}{\varphi} \wedge \mathbf{E}_1$	$\dfrac{\begin{array}{c}\vdots\\ \varphi \wedge \psi\end{array}}{\psi} \wedge \mathbf{E}_2$
$\vee$	$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\varphi \vee \psi} \vee \mathbf{I}_1$	$\dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\varphi \vee \psi} \vee \mathbf{I}_2$	$\dfrac{\begin{array}{ccc} & [\varphi]^i & [\psi]^j\\ \vdots & \vdots & \vdots\\ \varphi \vee \psi & \gamma & \gamma\end{array}}{\gamma} \vee \mathbf{E}, i, j$	
$\to$	$\dfrac{\begin{array}{c}[\varphi]^i\\ \vdots\\ \psi\end{array}}{\varphi \to \psi} \to \mathbf{I}, i$		$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \varphi & \varphi \to \psi\end{array}}{\psi} \to \mathbf{E}$	

Examples

$\vdash \varphi \to (\psi \to \varphi)$

$$\dfrac{\dfrac{[\varphi]^2}{\psi \to \varphi}\;1}{\varphi \to (\psi \to \varphi)}\;2$$

$\vdash (\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to (\psi \to \gamma)]^1 \qquad [\varphi]^2}{\psi \to \gamma} \qquad \dfrac{[\varphi \to \psi]^3 \qquad [\varphi]^2}{\psi}}{\gamma}}{\varphi \to \gamma}\;2}{(\varphi \to \psi) \to (\varphi \to \gamma)}\;3}{(\varphi \to (\psi \to \gamma)) \to (\varphi \to \psi) \to (\varphi \to \gamma)}\;1$$

Natural Deduction, NK_0 (cont.)

	Introdution	**Elimination**
$\neg$	$\dfrac{\begin{matrix}[\varphi]^i \\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\;\neg\mathbf{I}, i$	$\dfrac{\begin{matrix}\vdots \\ \varphi\end{matrix} \qquad \begin{matrix}\vdots \\ \neg\varphi\end{matrix}}{\psi}\;\neg\mathbf{E}$
$\neg\neg$	$\dfrac{\begin{matrix}\vdots \\ \varphi\end{matrix}}{\neg\neg\varphi}\;\neg\neg\mathbf{I}$	$\dfrac{\begin{matrix}\vdots \\ \neg\neg\varphi\end{matrix}}{\varphi}\;\neg\neg\mathbf{E}$

Example

$\vdash (\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\psi \to \neg\varphi]^1 \quad [\neg\psi]^2}{\neg\varphi} \quad \dfrac{[\neg\psi \to \varphi]^3 \quad [\neg\psi]^2}{\varphi}}{\bot}}{\neg\neg\psi}\,2}{\psi}}{(\neg\psi \to \varphi) \to \psi}\,3}{(\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)}\,1$$

Modus Tollens

$$\dfrac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}\,\mathbf{MT}$$

$$\dfrac{\dfrac{\dfrac{\varphi \to \psi \qquad [\varphi]^1}{\psi} \qquad \neg\psi}{\bot}}{\neg\varphi}\,1$$

Excluded middle

$$\frac{}{\varphi \vee \neg\varphi}\,\textbf{TE}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi \vee \neg\varphi)]^3 \qquad \dfrac{[\neg\varphi]^1}{\varphi \vee \neg\varphi}}{\bot}}{\neg\neg\varphi}\,1}{\varphi} \qquad \dfrac{\dfrac{[\neg(\varphi \vee \neg\varphi)]^3 \qquad \dfrac{[\varphi]^2}{\varphi \vee \neg\varphi}}{\bot}}{\neg\varphi}\,2}{\bot}}{\dfrac{\neg\neg(\varphi \vee \neg\varphi)}{\varphi \vee \neg\varphi}}\,3$$

Soundness and Completeness

Theorem

The system NK_0 is sound and complete for classic propositional logic: $\vdash^{NK_0} \varphi$ *iff* $\models \varphi$. *And* $\Gamma \vdash^{NK_0} \varphi$ *iff* $\Gamma \models \varphi$

Theorem (Decidability)

There is an algorithm which, given a finite set Γ of formulas and a formula φ, decides whether $\Gamma \vdash^{NK_0} \varphi$.

First Order Logic

Syntax

- $Var = \{x, y, \ldots, x_0, y_0, \ldots\}$ infinite set of variables
- logic connectives $\wedge$, $\vee$, $\neg$ e $\rightarrow$;
- quantifiers $\forall$ (universal) and $\exists$ (existential);
- parenthesis (and);
- a set $\mathcal{F}$ of *functional symbols* f, g, h, ...;
- a set $\mathcal{P}$ of *predicate symbols* P, Q, R, ...;
- an *arity function* $m : \mathcal{F} \cup \mathcal{R} \longrightarrow \mathbb{N}$

First Order Logic

Terms

The set $\mathcal{T}$ of Terms t, s, ... are inductively defined by:

- a variable is a term
- for every $f \in F$ of arity m, for all terms t_1, ..., t_m, $f(t_1, \ldots, t_m)$ is also a term

First Order Logic

Formulas

- if $t_1, \ldots, t_m$ are terms and $R \in \mathcal{P}$ of arity m then $R(t_1, \ldots, t_n)$ is an atomic formula.
- an atomic formula is a formula
- if φ is a formula, $\neg\varphi$ is a formula;
- if φ and ψ are formulas, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \rightarrow \psi)$ are *formulas*
- if φ is a formula and x a variable, then $\forall x\ \varphi$ and $\exists x\ \varphi$ are *formulas*.

Natural Deduction, *NK*: quantifiers rules

	Introduction	**Elimination**
$\forall$	$\dfrac{\begin{matrix}[v]\\ \vdots \\ \varphi[v/x]\end{matrix}}{\forall x\ \varphi}\ \forall\mathbf{I}\quad (a)$	$\dfrac{\forall x\ \varphi}{\varphi[t/x]}\ \forall\mathbf{E}\quad (b)$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}\ \exists\mathbf{I}\quad (b)$	$\dfrac{\exists x\ \varphi \qquad \begin{matrix}[v \quad \varphi[v/x]]\\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists\mathbf{E}\quad (c)$

(a) where v is a new variable

(b) where x is free for t in φ

(c) where v is a new variable and not in ψ

Example

$\neg\forall x\neg\varphi \rightarrow \exists x\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\forall x\neg\varphi]^3 \qquad \dfrac{\dfrac{\dfrac{[\neg\exists x\varphi]^2 \qquad \dfrac{[\varphi[u/x]]^1}{\exists x\varphi}}{\bot}}{\neg\varphi[u/x]}\;1}{\forall x\neg\varphi}}{\bot}}{\neg\neg\exists x\varphi}\;2}{\exists x\varphi}}{\neg\forall x\neg\varphi \rightarrow \exists x\varphi}\;3$$

Sequents

In each step of a tree derivation is not easy to know which are the open assumptions:

If φ depends on open assumptions $\varphi_1, \ldots, \varphi_k$:

$$\varphi_1 \wedge \ldots \wedge \varphi_k \rightarrow \varphi$$

Sequents

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$$

Meaning: $\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \psi_1 \vee \ldots \vee \psi_m$

Empty antecedent: $\psi_1 \vee \ldots \vee \psi_m$

Empty consequent: $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$

Both empty: $\bot$

NK_0 in *sequent form*

Γ (context) is a set of formulas

$$\frac{}{\Gamma,\varphi \Rightarrow \varphi}\textbf{Ax}$$

	Introdution	**Elimination**
$\wedge$	$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \wedge \textbf{I}$	$\frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi} \wedge \textbf{E}_1$ $\quad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi} \wedge \textbf{E}_2$
$\vee$	$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \vee \textbf{I}_1$ $\quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \vee \textbf{I}_2$	$\frac{\Gamma \Rightarrow \varphi \vee \psi \quad \Gamma,\varphi \Rightarrow \gamma \quad \Gamma,\psi \Rightarrow \gamma}{\Gamma \Rightarrow \gamma} \vee \textbf{E}$
$\rightarrow$	$\frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi\rightarrow\psi} \rightarrow\textbf{I}$	$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \varphi\rightarrow\psi}{\Gamma \Rightarrow \psi} \rightarrow\textbf{E}$
$\neg$	$\frac{\Gamma,\varphi \Rightarrow \bot}{\Gamma \Rightarrow \neg\varphi} \neg\textbf{I}$	$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \neg\varphi}{\Gamma \Rightarrow \psi} \neg\textbf{E}$
$\neg\neg$	$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \neg\neg\varphi} \neg\neg\textbf{I}$	$\frac{\Gamma \Rightarrow \neg\neg\varphi}{\Gamma \Rightarrow \varphi} \neg\neg\textbf{E}$

Deductions with *sequents*

Nodes of the derivation trees are *sequents* and $\vdash \Gamma \Rightarrow \varphi$ means $\Gamma \vdash \varphi$

$\vdash \Rightarrow \varphi \rightarrow (\psi \rightarrow \varphi)$

$$\cfrac{\cfrac{\varphi,\psi \Rightarrow \varphi}{\varphi \Rightarrow \psi\rightarrow\varphi}(\rightarrow\textbf{I})}{\Rightarrow \varphi\rightarrow(\psi\rightarrow\varphi)}(\rightarrow\textbf{I})$$

Weakening

If $\Gamma \Rightarrow \varphi$ is derivable, then for all $\Gamma' \supseteq \Gamma$, $\Gamma' \Rightarrow \varphi$ is derivable

NK in *sequent form*: quantifier rules

$$\frac{\Gamma \Rightarrow \varphi[v/x]}{\Gamma \Rightarrow \forall x\, \varphi}\ \forall\mathbf{I} \quad (a)$$

$$\frac{\Gamma \Rightarrow \forall x\, \varphi}{\Gamma \Rightarrow \varphi[t/x]}\ \forall\mathbf{E} \quad (b)$$

$$\frac{\Gamma \Rightarrow \varphi[t/x]}{\Gamma \Rightarrow \exists x\, \varphi}\ \exists\mathbf{I} \quad (b)$$

$$\frac{\Gamma \Rightarrow \exists x\, \varphi \qquad \Gamma, \varphi[v/x] \Rightarrow \psi}{\Gamma \Rightarrow \psi}\ \exists\mathbf{E} \quad (c)$$

(a) where v is a new variable, not in Γ

(b) where x is free for t in φ

(c) where v is a new variable, not in Γ and not in ψ

Gentzen *Sequent* calculi

Deductive systems introduced by Gentzen (1935) in order to obtain deduction normal forms.

Allows decidability algorithms without using completeness (semantics).

Modus ponens:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

given ψ, φ can be any formula...

Although normalisation can be obtained for NK_0, these systems are more structured and reveal the structure of the deductions:are the base of other *analytic deduction systems* as resolution and *tableaux*.

Two types of introduction rules: in the antecedent (**L**) and in the consequent (**R**).

Sequent calculus LK_0

$$\frac{}{\Gamma,\varphi \Rightarrow \Delta,\varphi}\,\mathbf{Ax}$$

$$\frac{\Gamma \Rightarrow \Delta,\varphi \qquad \Gamma',\varphi \Rightarrow \Delta'}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta'}\,\mathbf{Cut}$$

$$\frac{\Gamma,\varphi,\psi \Rightarrow \Delta}{\Gamma,\varphi \wedge \psi \Rightarrow \Delta}\wedge\mathbf{L} \qquad \frac{\Gamma \Rightarrow \Delta,\varphi \qquad \Gamma \Rightarrow \Delta,\psi}{\Gamma \Rightarrow \Delta,\varphi \wedge \psi}\wedge\mathbf{R}$$

$$\frac{\Gamma,\varphi \Rightarrow \Delta \qquad \Gamma,\psi \Rightarrow \Delta}{\Gamma,\varphi \vee \psi \Rightarrow \Delta}\vee\mathbf{L} \qquad \frac{\Gamma \Rightarrow \Delta,\varphi,\psi}{\Gamma \Rightarrow \Delta,\varphi \vee \psi}\vee\mathbf{R}$$

$$\frac{\Gamma \Rightarrow \varphi,\Delta \qquad \Gamma,\psi \Rightarrow \Delta}{\Gamma,\varphi\to\psi \Rightarrow \Delta}\to\mathbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \Delta,\psi}{\Gamma \Rightarrow \Delta,\varphi\to\psi}\to\mathbf{R}$$

$$\frac{\Gamma \Rightarrow \Delta,\varphi}{\Gamma,\neg\varphi \Rightarrow \Delta}\neg\mathbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\neg\varphi}\neg\mathbf{R}$$

$\vdash \Rightarrow \; ((p \to q) \to p) \to p$

$$\frac{\dfrac{\dfrac{p \Rightarrow p,q}{\Rightarrow p,(p\to q)}(\to\mathbf{R}) \qquad p \Rightarrow p}{(p\to q)\to p) \Rightarrow p}(\to\mathbf{L})}{\Rightarrow ((p\to q)\to p)\to p}(\to\mathbf{R})$$

$\vdash \Rightarrow \; (p \to (p \to q)) \to (p \to q)$

$$\frac{\dfrac{\dfrac{p \Rightarrow p,q \qquad \dfrac{p \Rightarrow p,q \qquad p,q \Rightarrow q}{(p\to q),p \Rightarrow q}(\to\mathbf{L})}{p\to(p\to q),p \Rightarrow q}(\to\mathbf{L})}{(p\to(p\to q) \Rightarrow (p\to q)}(\to\mathbf{R})}{\Rightarrow (p\to(p\to q))\to(p\to q)}(\to\mathbf{R})$$

NK_0 versus LK_0

Rules

NK_0 (in *sequents*)	LK_0
Axiom	Axiom
Introduction($\circ$**I**)	Introduction in the consequent ($\circ$**R**)
Elimination ($\circ$**E**)	Introduction in the antecedent ($\circ$**L**)

Subformula property

In a cut-free deduction of $\Gamma \Rightarrow \Delta$, all sequents are composed of subformulas of formulas in Γ, Δ only

Then is possible to obtain an algorithm that searches a proof from the root

Cut elimination

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma', \varphi \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\textbf{Cut}$$

φ cut formula

Theorem

(Hauptsatz) The deduction system LK_0, without cut rule, is sound and complete.
There is an algorithm that takes a deduction in LK_0, and turns it into a cut-free deduction of the same sequent.

Why this rule?:

- allows shorter deductions
- isolates the redundancy of the deductions
- is easier to obtain certain theoretical results
- there are systems that recover part of its functionality...preserving a normal form (*Tableaux* KE)

Idea of the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma,\varphi \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \varphi\to\psi}(\to\mathbf{R}) \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma,\psi \Rightarrow \gamma\end{array}}{\Gamma,\varphi\to\psi \Rightarrow \gamma}(\to\mathbf{L})}{\Gamma \Rightarrow \gamma}\,\mathbf{Cut}$$

Transformation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma,\varphi \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \psi}\,\mathbf{Cut} \qquad \begin{array}{c}\vdots\\ \Gamma,\psi \Rightarrow \gamma\end{array}}{\Gamma \Rightarrow \gamma}\,\mathbf{Cut}$$

- Transform applications of the cut rule in others with simple cut formulas
- Cut rules applications in upper nodes of the derivation tree
- Requires double induction: in the depth of the cut rule applications and in the complexity of the cut formula.

LK calculus: quantifier rules

$$\dfrac{\Gamma \Rightarrow \varphi[v/x], \Delta}{\Gamma \Rightarrow \forall x\, \varphi, \Delta}\,\forall\mathbf{R} \quad \text{(a)}$$

$$\dfrac{\Gamma, \varphi[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, \varphi \Rightarrow \Delta}\,\forall\mathbf{L} \quad \text{(b)}$$

$$\dfrac{\Gamma \Rightarrow \varphi[t/x], \Delta}{\Gamma \Rightarrow \exists x\, \varphi, \Delta}\,\exists\mathbf{R} \quad \text{(b)}$$

$$\dfrac{\Gamma, \varphi[v/x] \Rightarrow \Delta}{\Gamma, \exists x\, \varphi \Rightarrow \Delta}\,\exists\mathbf{L} \quad \text{(a)}$$

(a) where v is a new variable, not in Γ, Δ

(b) where x is free for t in φ

Methods of proof

The semantics of classical logic is based on the notion of *truth*. Each proposition is absolutely true or false.

Principle of the excluded middle: $p \lor \neg p$.

But that gives not much information...

> *There are two irrational numbers b and c, such that b^c is rational*

Proof by case analysis:

if $\sqrt{2}^{\sqrt{2}}$ is a rational number then we can take $b = c = \sqrt{2}$; otherwise take $b = \sqrt{2}^{\sqrt{2}}$ and $c = \sqrt{2}$.

But which are these values?...The problem is that the proof is not *constructive*

Intuitionism

- φ is true if we can prove it.
- φ is false if we can show that if we have a proof of φ we get a contradiction.

Informal constructive semantics of connective (*BHK*-interpretation)

- A construction of $\varphi \land \psi$ consists of a construction of φ and a construction of ψ
- A construction of $\varphi \lor \psi$ consists of a construction of φ or a construction of ψ
- A construction of $\varphi \to \psi$ is a method (function) transforming every construction of φ into a construction of ψ
- A construction of $\neg\varphi$ is a method transforming every construction of φ into a no existent object (there is no construction of $\bot$)

Intuitionism

Show that

$p \to \neg\neg p$ (or $p \to ((p \to \bot) \to \bot)$) is intuitionistically valid:

Given a proof of p, we can obtain a proof of $((p \to \bot) \to \bot)$*: Take a proof of* $(p \to \bot)$*. It is a method to transform proofs of p into proofs of* $\bot$*. Since we have a proof of p, we can obtain a proof of* $\bot$*.*

But

$\neg\neg p \to p$ is not intuicionistically valid: the fact of not having a proof of $\neg p$, does not allow to conclude that we have a proof of p...

In the same way, $p \vee \neg p$ is not valid!... in general, is nor possible to ensure that we have a proof of p or of $\neg p$.

Intuicionistic logic is obtained from classic logic by eliminating the rule $\neg\neg\mathbf{E}$ in the NK_0 system...

Structural rules for *sequents*

Allow the explicit manipulation of assumptions in a proof.

In a *sequent* $\Gamma \Rightarrow \Delta$, Γ and Δ are lists of formulas.

Weakening

$$\frac{\Gamma \Rightarrow \Delta}{\varphi,\Gamma \Rightarrow \Delta}WL \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\varphi}WR$$

Contraction

$$\frac{\varphi,\varphi,\Gamma \Rightarrow \Delta}{\varphi,\Gamma \Rightarrow \Delta}CL \qquad \frac{\Gamma \Rightarrow \Delta,\varphi,\varphi}{\Gamma \Rightarrow \Delta,\varphi}CR$$

Permutation

$$\frac{\Lambda,\varphi,\psi,\Gamma \Rightarrow \Delta}{\Lambda,\psi,\varphi,\Gamma \Rightarrow \Delta}XL \qquad \frac{\Gamma \Rightarrow \Delta,\varphi,\psi,\Lambda}{\Gamma \Rightarrow \Delta,\psi,\varphi,\Lambda}XR$$

Intuitionistic Natural Deduction NJ_0

In intuitionistic logic, in a sequent $\Gamma \Rightarrow \Delta$, Δ has at most one formula.

$$\frac{}{\varphi \Rightarrow \varphi}\,\mathbf{Ax}$$

$$\frac{\Gamma \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}\,W \qquad \frac{\Gamma,\varphi,\varphi \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}\,C \qquad \frac{\Gamma,\varphi,\psi,\Delta \Rightarrow \varphi'}{\Gamma,\psi,\varphi,\Delta \Rightarrow \varphi'}\,X$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \varphi \wedge \psi}\,\wedge\,\mathbf{I} \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi}\,\wedge\,\mathbf{E}_1 \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}\,\wedge\,\mathbf{E}_2$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\,\vee\,\mathbf{I}_1 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\,\vee\,\mathbf{I}_2 \qquad \frac{\Gamma \Rightarrow \varphi \vee \psi \qquad \Delta,\varphi \Rightarrow \gamma \qquad \Delta,\psi \Rightarrow \gamma}{\Gamma,\Delta \Rightarrow \gamma}\,\vee\,\mathbf{E}$$

$$\frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\,{\to}\mathbf{I} \qquad \frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \varphi \to \psi}{\Gamma,\Delta \Rightarrow \psi}\,{\to}\mathbf{E}$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\,\bot\mathbf{E}$$

Intuitionistic Natural Deduction NJ_0

Negation can be defined by $\neg\varphi = \varphi \to \bot$

$\vdash_{NJ_0} \Rightarrow \ p \to \neg\neg p$

$$\cfrac{\cfrac{\cfrac{p \Rightarrow p \qquad p \to \bot \Rightarrow p \to \bot}{p, p \to \bot \Rightarrow \bot}}{p \Rightarrow (p \to \bot) \to \bot}}{\Rightarrow p \to ((p \to \bot) \to \bot)}$$

$\not\vdash_{NJ_0} \Rightarrow \ \neg\neg p \to p$...

de Morgan laws are not valid any more...

$\Rightarrow \ p \vee \neg p$ is not derivable

The last rule should be $\vee$ **I**. Then $\Rightarrow p$ or $\Rightarrow \neg p$ are derivable. $\Rightarrow p$ is not derivable. For $\Rightarrow \neg p$, i.e., $\Rightarrow p \to \bot$, the last rule must be $\to$ **I**, but no rule has a conclusion $p \Rightarrow \bot$.

Intuitionistic *Sequent* Calculus, LJ_0

$$\frac{}{\varphi \Rightarrow \varphi}\,\mathbf{Ax}$$

$$\frac{\Gamma \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}\,W \quad \frac{\Gamma,\varphi,\varphi \Rightarrow \varphi'}{\Gamma,\varphi \Rightarrow \varphi'}\,C \quad \frac{\Gamma,\varphi,\psi,\Delta \Rightarrow \varphi'}{\Gamma,\psi,\varphi,\Delta \Rightarrow \varphi'}\,X \quad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\,\bot$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta,\varphi \Rightarrow \varphi'}{\Gamma,\Delta \Rightarrow \varphi'}\,\mathbf{Cut}$$

$$\frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma,\varphi \wedge \varphi' \Rightarrow \psi}\,\wedge\mathbf{L}_1 \quad \frac{\Gamma,\varphi' \Rightarrow \psi}{\Gamma,\varphi \wedge \varphi' \Rightarrow \psi}\,\wedge\mathbf{L}_2 \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \varphi \wedge \psi}\,\wedge\mathbf{R}$$

$$\frac{\Gamma,\varphi \Rightarrow \psi \qquad \Delta,\varphi' \Rightarrow \psi}{\Gamma,\Delta,\varphi \vee \varphi' \Rightarrow \psi}\,\vee\mathbf{L} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\,\vee\mathbf{R}_2 \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\,\vee\mathbf{R}_1$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta,\psi \Rightarrow \varphi'}{\Gamma,\Delta,\varphi\to\psi \Rightarrow \varphi'}\,\to\mathbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi\to\psi}\,\to\mathbf{R}$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma,\neg\varphi \Rightarrow \bot}\,\neg\mathbf{L} \qquad \frac{\Gamma,\varphi \Rightarrow \bot}{\Gamma \Rightarrow \neg\varphi}\,\neg\mathbf{R}$$

Intuitionistic *Sequent* Calculus, LJ_0

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\neg p \Rightarrow \neg p}\,(\mathrm{Ax})}{\neg p, \neg\neg p \Rightarrow \bot}\,(\neg L)}{\neg p \Rightarrow \neg\neg\neg p}\,(\neg R)}{\Rightarrow \neg p \to \neg\neg\neg p}\,(\to R)$$

Proposition

$\Gamma \vdash_{LJ_0} \varphi$ *iff* $\Gamma \vdash_{NJ_0} \varphi$

Idea of the proof

($\Rightarrow$)

In the mapping $\mathcal{N}$ from proofs of LJ_0 to proofs in NJ_0, a natural deduction NJ_0 version of the cut rule is used:

$$\frac{\Gamma \;\Rightarrow\; \varphi \qquad \Delta,\varphi \;\Rightarrow\; \varphi'}{\Gamma,\Delta \;\Rightarrow\; \varphi'}\, Subs$$

Which is a derivable extension:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \varphi\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Delta,\varphi \Rightarrow \varphi'\end{array}}{\Delta \Rightarrow \varphi \to \varphi'}\to I}{\Gamma,\Delta \;\Rightarrow\; \varphi'}\to E$$

Idea of the proof

($\wedge L_1$)

$$\dfrac{\dfrac{\pi}{\Gamma,\varphi \Rightarrow \varphi'}}{\Gamma,\varphi \wedge \psi \Rightarrow \varphi'}$$

is mapped to:

$$\dfrac{\dfrac{\overline{\varphi \wedge \psi \Rightarrow \varphi \wedge \psi}\,(Ax)}{\varphi \wedge \psi \Rightarrow \varphi}\wedge E \qquad \dfrac{N(\pi)}{\Gamma,\varphi \Rightarrow \varphi'}}{\Gamma,\varphi \wedge \psi \Rightarrow \varphi'}\, Subs$$

($\to L$)

$$\dfrac{\dfrac{\pi}{\Gamma \Rightarrow \varphi} \qquad \dfrac{\pi'}{\Delta,\psi \Rightarrow \varphi'}}{\Gamma,\Delta,\varphi\to\psi \Rightarrow \varphi'}$$

is mapped to :

$$\dfrac{\dfrac{\overline{\varphi\to\psi \Rightarrow \varphi\to\psi}\,(Ax) \qquad \dfrac{N(\pi)}{\Gamma \Rightarrow \varphi}}{\Gamma,\varphi\to\psi \Rightarrow \psi}\to E \qquad \dfrac{N(\pi')}{\Delta,\psi \Rightarrow \varphi'}}{\Gamma,\Delta,\varphi\to\psi \Rightarrow \varphi'}\, Subs$$

etc...

Semantics of intuitionistic logic

The deduction systems NJ_0 and LJ_0 are sound for classic truth valued semantics but they are not complete...

> *There is no finite truth table semantics for intuitionistic logic that is sound and complete*

Boolean algebras can be modified for intuitionistic logic (Heyting algebras), by introducing the notion of partiality.

The same can be done with notion of *possible worlds*.

Kripke semantics

A Kripke frame $\mathcal{F}$

is a structure $(X, \leq, \models)$ where:

- $(X, \leq)$ is a partial order
- $\models$ is a binary relation in $X \times \mathcal{V}_{Prop}$ such that for all $x, y \in X$, if $x \models p$ and $x \leq y$ then $y \models p$.

Semantics

If $x \models p$, p is forced at x.

$\models$ extends to the set of formulae:

$x \models \varphi \wedge \psi$ iff $x \models \varphi$ and $x \models \psi$

$x \models \varphi \vee \psi$ iff $x \models \varphi$ or $x \models \psi$

$x \models \varphi \rightarrow \psi$ iff $\forall y$, $x \leq y$, $y \models \varphi$ then $y \models \psi$

$x \models \neg\varphi$ iff $\forall y$, $x \leq y$, $y \not\models \varphi$

$x \not\models \bot$, $\forall x$

Kripke semantics

$x \models \Gamma \ \Rightarrow \ \psi$ iff $\forall \varphi \in \Gamma$, $x \models \varphi$, then $x \models \psi$.

A formula φ is forced in $\mathcal{F}$ if every $x \in X$ forces φ

φ is intuitionistically valid if it is forced in every frame $\mathcal{F}$

Theorem (Monotonicity)

If $x \models \varphi$ and $x \leq y$ then $y \models \varphi$

Proposition

$x \models \varphi \rightarrow \bot$ *iff* $x \models \neg\varphi$

Proposition

$x \models \neg\neg\varphi$ *iff* $\forall y, x \leq y, \exists u, y \leq u$ *such that* $u \models \varphi$

Examples

- $\varphi \rightarrow \varphi$ is intuitionistically valid
  $x \models \varphi \rightarrow \varphi$ iff $\forall y$, $x \leq y$, $y \models \varphi$ then $y \models \varphi$
- $\varphi \rightarrow \neg\neg\varphi$ is intuitionistically valid
  $x \models \varphi \rightarrow \neg\neg\varphi$ iff $\forall y$, $x \leq y$, $y \models \varphi$ then $y \models \neg\neg\varphi$. And $y \models \neg\neg\varphi$ if $\forall u$, $y \leq u$,$\exists v$, $u \leq v$ and $v \models \varphi$. The result follows by transitivity of $\leq$ and monotonicity.
- $\neg\neg\varphi \rightarrow \varphi$ is not intuitionistically valid
  $x \models \neg\neg\varphi \rightarrow \varphi$, iff $\forall y$, $x \leq y$, $y \models \neg\neg\varphi$ then $y \models \varphi$. I.e., $\forall u$, $y \leq u$, $\exists v$, $u \leq v$, $v \models \varphi$ implies that $y \models \varphi$. For instance, take $(\{0, 00\}, (0 \leq 00), \models)$ with $00 \models p$.
- $p \ \vee \ \neg p$ is not intuitionistically valid.

Kripke semantics

Theorem (Soundness)

If $\Gamma \Rightarrow \varphi$ *is derivable in* NJ_0 *then* $\Gamma \Rightarrow \varphi$ *is intuitionistically valid.*

Theorem (Completeness)

If $\Gamma \Rightarrow \varphi$ *is intuitionistically valid, then* $\vdash_{NJ_0} \Gamma \Rightarrow \varphi$.

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