

Foundations of the PF relational calculus

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DI/UM, 2008

Recalling...

Monotonicity:

All operations are monotonic, eg.

$$\frac{R \subseteq S \quad T \subseteq U}{(R \cdot T) \subseteq (S \cdot U)}$$

$$\frac{R \subseteq S}{R^\circ \subseteq S^\circ}$$

Composition:

- Composition is associative:
- Identity:
- Empty relation:

$$R \cdot (S \cdot T) = (R \cdot S) \cdot T$$

$$R \cdot id = id \cdot R = R$$

$$R \cdot \perp = \perp \cdot R = \perp$$

Recalling...

Pointfree Relational Equality:

- **Cyclic inclusion** (“ping-pong”) rule:

$$R = S \equiv R \subseteq S \wedge S \subseteq R$$

- **Indirect equality** rules:

$$\begin{aligned} R = S &\equiv \langle \forall X :: (X \subseteq R \equiv X \subseteq S) \rangle \\ &\equiv \langle \forall X :: (R \subseteq X \equiv S \subseteq X) \rangle \end{aligned}$$

Relational algebra: converse

Properties:

$$^{\circ}\text{-universal}: \quad X^{\circ} \subseteq Y \equiv X \subseteq Y^{\circ} \quad (1)$$

$$^{\circ}\text{-monotonicity}: \quad R \subseteq S \equiv R^{\circ} \subseteq S^{\circ} \quad (2)$$

Then:

$$\text{Involution} : \quad (R^{\circ})^{\circ} = R \quad (3)$$

$$\text{Contravariance} : \quad (R \cdot S)^{\circ} = S^{\circ} \cdot R^{\circ} \quad (4)$$

These can be proved from $^{\circ}\text{-universal}$ by (elegant) indirect proofs (cf. exercises later on):

Relation algebra: meet and join

Properties:

$$\cap\text{-universal:} \quad X \subseteq (R \cap S) \equiv (X \subseteq R) \wedge (X \subseteq S) \quad (5)$$

$$\cup\text{-universal:} \quad R \cup S \subseteq X \equiv (R \subseteq X) \wedge (S \subseteq X) \quad (6)$$

Then

- Converse distributes over \cap :

$$(R \cap S)^{\circ} = R^{\circ} \cap S^{\circ} \quad (7)$$

- Converse distributes over \cup :

$$(R \cup S)^{\circ} = R^{\circ} \cup S^{\circ} \quad (8)$$

(sample calculational proof follows)

Relation algebra: proofs by calculation

Exercise 1: Complete the following calculation by indirect equality:

$$\begin{aligned} & X \subseteq R^\circ \cap S^\circ \\ \equiv & \quad \{ \dots \} \\ & (X \subseteq R^\circ) \wedge (X \subseteq S^\circ) \\ \equiv & \quad \{ \dots \} \\ & (X^\circ \subseteq R) \wedge (X^\circ \subseteq S) \\ \equiv & \quad \{ \dots \} \\ & X^\circ \subseteq R \cap S \\ \equiv & \quad \{ \dots \} \\ & X \subseteq (R \cap S)^\circ \\ \therefore & \quad \{ \text{indirection} \} \\ & R^\circ \cap S^\circ = (R \cap S)^\circ \end{aligned}$$

Relational calculus: functions

Shunting rules:

$$f \cdot R \subseteq S \equiv R \subseteq f^\circ \cdot S \quad (9)$$

$$R \cdot f^\circ \subseteq S \equiv R \subseteq S \cdot f \quad (10)$$

Equality:

$$f \subseteq g \equiv f = g \equiv f \supseteq g \quad (11)$$

“Cyclic inclusion” calculation of the equality rule (11) follows.

Proof of functional equality

$$\begin{aligned} & f \subseteq g \\ \equiv & \quad \{ \text{identity} \} \\ & f \cdot id \subseteq g \\ \equiv & \quad \{ \text{shunting on } f \} \\ & id \subseteq f^\circ \cdot g \\ \equiv & \quad \{ \text{shunting on } g \} \\ & id \cdot g^\circ \subseteq f^\circ \\ \equiv & \quad \{ \text{converses; identity} \} \\ & g \subseteq f \end{aligned}$$

Adding structure to the calculus

Note a recurrent **pattern** in several laws above:

$$\begin{array}{lcl} \underbrace{X^\circ}_{f \ X} \subseteq Y & \equiv & X \subseteq \underbrace{Y^\circ}_{g \ Y} \\ \underbrace{(h \cdot) X}_{f \ X} \subseteq Y & \equiv & X \subseteq \underbrace{(h^\circ \cdot) Y}_{g \ Y} \\ \underbrace{X(\cdot h^\circ)}_{f \ X} \subseteq Y & \equiv & X \subseteq \underbrace{Y(\cdot h)}_{g \ Y} \end{array}$$

as well as in

$$\underbrace{(d \times) q}_{f \ q} \leq n \quad \equiv \quad q \leq \underbrace{n(/d)}_{g \ n}$$

where $(/d)$ denotes integral division (in \mathbb{N}_0).

Back to the primary school desk

The **integral division** algorithm

$$\begin{array}{r} 7 \\ 1 \overline{) 3} \end{array} \quad 2 \times 3 + 1 = 7 \quad , \text{ "ie." } \quad 3 = 7/2$$

However

$$\begin{array}{r} 7 \\ 3 \overline{) 2} \end{array} \quad 2 \times 2 + 3 = 7 \quad \wedge \quad 2 \neq 7/2$$

$$\begin{array}{r} 7 \\ 5 \overline{) 1} \end{array} \quad 2 \times 1 + 5 = 7 \quad \wedge \quad 1 \neq 7/2$$

In fact:

$$\begin{array}{r} n \\ r \overline{) d} \end{array} \quad q = n/d \equiv d \times q + r = n$$

provided q is the
largest such q (r
smallest)

Back to the primary school desk

So:

- Quotient is a supremum:

$$\begin{aligned}n/d &= \langle \bigvee q :: \langle \exists r :: d \times q + r = n \rangle \rangle \\ &= \langle \bigvee q :: d \times q \leq n \rangle\end{aligned}$$

- Maths teachers tell: *it takes a while before children master the “ \bigvee semantics”!*
- What about you? Can you easily reason about n/d in this format?
- Challenge: Try and prove $(n/m)/d = n/(d \times m)$.

Proposed alternative: **al-djabr** rule

$$q \times d \leq n \equiv q \leq n/d$$

“universal”
property of
integral division

(12)

“Al-djabr” calculation instead

$$\begin{aligned} & q \leq (n/m)/d \\ \equiv & \quad \{ \text{“al-djabr” (12)} \} \\ & q \times d \leq n/m \\ \equiv & \quad \{ \text{“al-djabr” (12)} \} \\ & (q \times d) \times m \leq n \\ \equiv & \quad \{ \times \text{ is associative} \} \\ & q \times (d \times m) \leq n \\ \equiv & \quad \{ \text{“al-djabr” (12)} \} \\ & q \leq n/(d \times m) \\ \therefore & \quad \{ \text{indirection} \} \\ & (n/m)/d = n/(d \times m) \end{aligned}$$

(Generic) indirect equality

Note the use of **indirect equality** rule

$$(q \leq x \equiv q \leq y) \equiv (x = y)$$

in fact valid for \leq **any** partial order.

Exercise 2: Derive from (12) the two *cancellation* laws

$$q \leq (q \times d)/d \quad (13)$$

$$(n/d) \times d \leq n \quad (14)$$

and *reflexion* law:

$$n/d \geq 1 \equiv d \leq n \quad (15)$$



Galois connections

n/d is an example of operation involved in a **Galois** connection:

$$\underbrace{q \times d}_{f \ q} \leq n \quad \equiv \quad q \leq \underbrace{n/d}_{g \ n}$$

In general, for **preorders** (A, \leq) and (B, \sqsubseteq) and

$$\begin{array}{ccc} & \xrightarrow{g} & \\ (A, \leq) & & (B, \sqsubseteq) \\ & \xleftarrow{f} & \end{array} \quad (16)$$

(f, g) are *Galois connected* iff. . .

Galois adjoints

$$\underbrace{f}_{\text{lower adjoint}} b \leq a \equiv b \sqsubseteq \underbrace{g}_{\text{upper adjoint}} a$$

that is

$$f^{\circ} \cdot \leq = \sqsubseteq \cdot g$$

Remarks:

- Galois (connected) adjoints enjoy a number of interesting **generic** properties
- *Very elegant* — **calculational** — way of performing *equational* reasoning (including *logical* deduction)

Basic properties

Cancellation:

$$(f \cdot g)a \leq a \quad \text{and} \quad b \sqsubseteq (g \cdot f)b$$

Distribution (in case of lattice structures):

$$\begin{aligned} f(a \sqcup a') &= (f a) \vee (f a') \\ g(b \wedge b') &= (g b) \sqcap (g b') \end{aligned}$$

Conversely,

- If f distributes over \sqcup then it has an upper adjoint g ($f^\#$)
- If g distributes over \wedge then it has a lower adjoint f (g^\flat)

Other properties

If (f, g) are Galois connected,

- $f(g)$ **uniquely** determines $g(f)$ — thus the $_b$, $_h$ notations
- f and g are **monotonic**
- (g, f) are also Galois connected — just **reverse** the orderings
- $f = f \cdot g \cdot f$ and $g = g \cdot f \cdot g$

etc

Summary

$(f\ b) \leq a \equiv b \sqsubseteq (g\ a)$		
Description	$f = g^b$	$g = f^\sharp$
Definition	$f\ b = \bigwedge \{a : b \sqsubseteq g\ a\}$	$g\ a = \bigsqcup \{b : f\ b \leq a\}$
Cancellation	$f(g\ a) \leq a$	$b \sqsubseteq g(f\ b)$
Distribution	$f(b \sqcup b') = (f\ b) \vee (f\ b')$	$g(a' \sqcap a) = (g\ a') \sqcap (g\ a)$
Monotonicity	$b \sqsubseteq b' \Rightarrow f\ b \leq f\ b'$	$a \leq a' \Rightarrow g\ a \sqsubseteq g\ a'$

In the sequel we will re-interpret the relational operators we've seen so far as Galois adjoints.

Converse

$(f \ X) \subseteq Y \equiv X \subseteq (g \ Y)$			
Description	$f = g^b$	$g = f^\sharp$	Obs.
converse	$(-)^{\circ}$	$(-)^{\circ}$	$bR^{\circ}a \equiv aRb$

Thus:

Cancellation $(R^{\circ})^{\circ} = R$

Monotonicity $R \subseteq S \equiv R^{\circ} \subseteq S^{\circ}$

Distributions $(R \cap S)^{\circ} = R^{\circ} \cap S^{\circ}, (R \cup S)^{\circ} = R^{\circ} \cup S^{\circ}$

Functions

$(f \ X) \subseteq Y \equiv X \subseteq (g \ Y)$			
Description	$f = g^b$	$g = f^\sharp$	Obs.
shunting rule	$(h \cdot)$	$(h^\circ \cdot)$	NB: h is a function
“converse” shunting rule	$(\cdot h^\circ)$	$(\cdot h)$	NB: h is a function

Consequences:

Functional equality: $h \subseteq g \equiv h = k \equiv h \supseteq k$

Functional division: $h^\circ \cdot R = h \setminus R$

Question: what does $h \setminus R$ mean?

Relational division

$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$			
Description	$f = g^b$	$g = f^\sharp$	Obs.
left-division	$(R \cdot)$	$(R \setminus \)$	left-factor
right-division	$(\cdot R)$	$(\ / R)$	right-factor

that is,

$$R \cdot X \subseteq Y \equiv X \subseteq R \setminus Y \quad (17)$$

$$X \cdot R \subseteq Y \equiv X \subseteq Y / R \quad (18)$$

Immediate: $(R \cdot)$ and $(\cdot R)$ distribute over union:

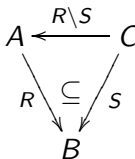
$$R \cdot (S \cup T) = (R \cdot S) \cup (R \cdot T)$$

$$(S \cup T) \cdot R = (S \cdot R) \cup (T \cdot R)$$

Some intuition about relational division operators follows.

Relational (left) division

Left division abstracts a (pointwise) universal quantification


$$a(R \setminus S)c \equiv \langle \forall b : b R a : b S c \rangle \quad (19)$$

Example:

$b R a$ = flight b carries passenger a

$b S c$ = flight b belongs to air-company c

$a (R \setminus S) c$ = passenger a is faithful to company c , that is, (s)he only flies company c .

Relational (right) division

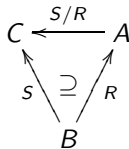
By taking converses we arrive at $S / R = (R^\circ \setminus S^\circ)^\circ$:

$$\begin{aligned} & X \subseteq S / R \\ \equiv & \quad \{ \text{Galois connection } ((\cdot R), (/R)) \} \\ & X \cdot R \subseteq S \\ \equiv & \quad \{ \text{converses} \} \\ & R^\circ \cdot X^\circ \subseteq S^\circ \\ \equiv & \quad \{ \text{Galois connection } ((R\cdot), (R\backslash)) \} \\ & X^\circ \subseteq R^\circ \setminus S^\circ \\ \equiv & \quad \{ \text{converses} \} \\ & X \subseteq (R^\circ \setminus S^\circ)^\circ \\ \therefore & \quad \{ \text{indirection} \} \\ & S / R = (R^\circ \setminus S^\circ)^\circ \end{aligned}$$

Relational (right) division

Therefore:

$$\begin{aligned} & c(S / R)a \\ \equiv & \quad \{ \text{above} \} \\ & a(R^\circ \setminus S^\circ)c \\ \equiv & \quad \{ (19) \} \\ & \langle \forall b : b R^\circ a : b S^\circ c \rangle \\ \equiv & \quad \{ \text{converses} \} \\ & \langle \forall b : a R b : c S b \rangle \end{aligned}$$



Domain and range

$(f \ X) \subseteq Y \equiv X \subseteq (g \ Y)$			
Description	$f = g^b$	$g = f^\sharp$	Obs.
domain	δ	$(\top \cdot)$	lower \subseteq restricted to coreflexives
range	ρ	$(\cdot \top)$	lower \subseteq restricted to coreflexives

Thus

$$\delta R \subseteq \Phi \equiv R \subseteq R \cdot \Phi \quad (20)$$

$$\rho R \subseteq \Phi \equiv R \subseteq \Phi \cdot \top \quad (21)$$

etc.

Domain and split

The following fact holds:

$$\langle R, S \rangle^\circ \cdot \langle X, Y \rangle = (R^\circ \cdot X) \cap (S^\circ \cdot Y)$$

Corollary:

$$\delta R = \ker \langle id, R \rangle$$

Another consequence of the fact above:

$$\ker R \subseteq \ker (S \cdot R) \iff S \text{ entire}$$

Corollary:

$$\ker R \subseteq \ker (f \cdot R)$$

Modular law

Dedekind's rule, also known as the **modular law**:

$$R \cdot S \cap T \subseteq R \cdot (S \cap R^\circ \cdot T)$$

cf. analogy with $ab + c \leq a(b + a^{-1}c)$. Dually (apply converses and rename):

$$(R \cdot S) \cap T \subseteq (R \cap (T \cdot S^\circ)) \cdot S$$

Symmetrical equivalent statement:

$$(R \cdot S) \cap T \subseteq (R \cap (T \cdot S^\circ)) \cdot (S \cap (R^\circ \cdot T))$$

= “weak right-distribution of meet over composition”.