Program Semantics, **Verification**, and Construction

Type Systems and Logics

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Proof Checking

- Proof checking consists of the automated verification of mathematical theories.
  - First one formalizes within a given logic the underlying primitive notions, the definitions, the axioms and the proofs;
  - and then the definitions are checked for their well-formedness and the proofs for their correctness.

  In this way mathematics is represented on a computer and also a hight degree of reliability is obtained.
- Once the theory is formalized, its correctness can be verified by the **proof-checker** (which is a small program).
- To help in the formalization process there exists an interactive **proof-development system**.
- Proof-checker and proof-development systems are usually combined in what is called a **proof-assistant**.

Proof-assistants

In a proof-assistant, after formalizing the primitive notions of the theory (under study), the user develops the proofs interactively by means of (proof) tactics, and when a proof is finished a "proof-term" is created. This proof-term closely corresponds to a standard mathematical proof (in natural deduction style).

Machine assisted theorem proving:

- helps to deal with large problems;
- prevents us from overseeing details;
- does the bookkeeping of the proofs.

Proof-assistants based on type theory present a general specification language to define mathematical notions and formulas. Moreover, it allows to construct algorithms and proofs as first class citizens.

Proof checking mathematical statements

- Mathematics is usually presented in an informal but precise way.

> In situation Γ we have A.
> Proof. p. QED

- In Logic Γ, A become formal objects and proofs can be formalized as a derivation tree (following some precisely given set of rules).

> $\Gamma \vdash_L A$
> Proof. p. QED

Types in logic

- The connection of type theory to logic is via the proposition-as-types principle that establishes a precise relation between intuitionistic logic and computation.
- Intuitionistic logic is based on the notion of proof – a proposition is true when we can provide a constructive proof of it. On this basis:
  - a proposition A can be seen as a type (the type of its proofs);
  - and a proof of A as an object of type A.

Hence: A is **provable** $\Leftrightarrow$ A is **inhabited**

Therefore, the formalization of mathematics in type theory becomes

$\Gamma \vdash_T p : A$ which is equivalent to $\mathrm{Type}_\Gamma(p) = A$

So, proof checking boils down to **type checking**.

Type-theoretic notions for proof-checking

In the practice of an interactive proof assistant based on type theory, the user types in tactics, guiding the proof development system to construct a proof-term. At the end, this term is type checked and the type is compared with the original goal.

In connection to proof checking there are some decidability problems:

Type Checking Problem (TCP) $\Gamma \vdash_T M : A \quad ?$

Type Synthesis Problem (TSP) $\Gamma \vdash_T M : ?$

Type Inhabitation Problem (TIP) $\Gamma \vdash_T ? : A$

TIP is usually undecidable for type theories of interest.

TCP and TSP are decidable for a large class of interesting type theories.

The reliability of machine checked proofs

- **Why would one believe a system that says it has verified a proof ?**

  The proof checker should be a very small program that can be verified by hand, giving the highest possible reliability to the proof checker.

- **de Bruijn criterion**

  A proof assistant satisfies the de Bruijn criterion if it generates proof-objects (of some form) that can be checked by an 'easy' algorithm.

Proof-objects may be large but they are self-evident. This means that a small program can verify them. The program just follows whether locally the correct steps are being made.

Type-theoretic approach to interactive theorem proving

provability of formula A	$\Longleftrightarrow$	inhabitation of type A
proof checking	$\Longleftrightarrow$	type checking
interactive theorem proving	$\Longleftrightarrow$	interactive construction of a term of a given type

So, decidability of type checking is at the core of the type-theoretic approach to theorem proving.

Examples of proof assistants based on type theory

The first systems of proof checking (type checking) based on the propositions-as-types principle were the systems of the **AUTOMATH** project.

Modern proof assistants aggregate to the proof checker a proof-development system for helping the user to develop the proofs interactively.

We can mention as examples of proof assistants, the systems:

- **Coq** , based on the Calculus of Inductive Constructions
- **Lego** , based on the Extended Calculus of Constructions
- **Alf** and **Agda** , based on Martin-Löf 's type theory
- **Nuprl** , based on extensional Martin-Löf 's type theory

Type Systems and Logics

Intuitionistic (constructive) logic

- A proof of $A \supset B$ is a method that transforms a proof of A into a proof of B.
- A proof of $A \wedge B$ is a pair (p, q) such that p is a proof of A and q is a proof of B.
- A proof of $A \vee B$ is a pair (b, p) where b is either 0 or 1 and, if $b{=}0$ then p is a proof of A; if $b{=}1$ then p is a proof of B.
- There is no proof of $\bot$, the false proposition.
- Negation $\neg A$ is defined as $A \supset \bot$.
- A proof of $\forall x \in X.\, P\,x$ is a method p that transforms every element $a \in X$ into a proof of Pa.
- A proof of $\exists x \in X.\, P\,x$ is a pair (a, p) such that $a \in X$ and p is a proof of Pa.

Propositions as types

A proposition A is interpreted as the collection of its proofs, represented by $[A]$.

So, according to the intuitionistic interpretation of the logical connectives one has

$$\begin{array}{rcl} [A \supset B] & = & [A] \to [B] \\ [A \wedge B] & = & [A] \times [B] \\ [A \vee B] & = & [A] \uplus [B] \\ [\bot] & = & \emptyset \\ [\forall x \in X.\, Px] & = & \Pi\, x{:}X.\, [Px] \\ [\exists x \in X.\, Px] & = & \Sigma\, x{:}X.\, [Px] \end{array}$$

where

$$\begin{array}{rcl} P \to Q & = & \{f \mid \forall p{:}P.\, f(p) : Q\} \\ P \times Q & = & \{(p, q) \mid p{:}P \ \text{ and } \ q{:}Q\} \\ P \uplus Q & = & \{(0, p) \mid p{:}P\} \bigcup \{(1, q) \mid q{:}Q\} \\ \Pi\, x{:}A.\, Bx & = & \left\{f : (A \to \textstyle\bigcup_{x:A} Bx) \mid \forall a{:}A.\, (fa : Ba)\right\} \\ \Sigma\, x{:}A.\, Bx & = & \{(a, p) \mid a{:}A \ \text{ and } \ p{:}(Ba)\} \end{array}$$

Example

Let X be a set and R be a binary relation on X. Now, consider the following lemma:

If $\forall x, y \in X.\ Rxy \supset \neg Ryx$ then $\forall x \in X.\ \neg Rxx$.

How can this be formalized ?

We have two universes Set and Prop

- a term X of type Set is a type that represents a domain of the logic;
- a term A : Prop is a type that represents a proposition of the logic;
- a predicate on X is represented by a term $P : X \to$ Prop

  $t : X$ satisfies the predicate P iff the type $(P\,t)$ is inhabited
  (i.e., there is a proof-term of type $(P\,t)$)

- a binary relation over X is represented by a term $R : X \to X \to$ Prop.

Example (cont.)

The collection of binary relations over X is represented as $X \to X \to$ Prop.

So, to represent the notion of (polymorphic) binary relation one has to abstract over the domains.

Let us define $\quad \mathsf{Rel} := \lambda X : \mathsf{Set}.\ X \to X \to \mathsf{Prop}$

Definitions are formal constructions in type theory with a computational rule associated, called **δ-reduction** by which definitions are unfolded.

$$\mathsf{D} \to_\delta M \qquad \text{if} \quad D := M$$

Anti-symmetry and irreflexivity can also be define as follows

$$\begin{array}{rcl} \mathsf{AntiSym} & := & \lambda X : \mathsf{Set}.\ \lambda R : (\mathsf{Rel}X).\ \forall x, y : X.\ Rxy \supset (Ryx \supset \bot) \\ \mathsf{Irrefl} & := & \lambda X : \mathsf{Set}.\lambda R : (\mathsf{Rel}X).\ \forall x : X.\ Rxx \supset \bot \end{array}$$

Note that $\neg A$ is defined as $A \supset \bot$ where $\bot$ is the empty type (the false proposition).

Example (cont.)

By δ and β-reductions we find that for X : Set and $Q : X \to X \to$ Prop

$$\begin{array}{lll} (\text{Rel } X) & =_{\delta\beta} & X \to X \to \text{Prop} \\ (\text{AntiSym } XQ) & =_{\delta\beta} & \forall x, y : X.\, Qxy \supset (Qyx \supset \bot) \\ (\text{Irrefl } XQ) & =_{\delta\beta} & \forall x : X.\, Qxx \supset \bot \end{array}$$

Here we have a **dependent type**, i.e., a type of functions f where the range-set depends on the input value.

The type of this kind of functions is $f : \Pi x : A.\, B$, the product of a family $\{Bx\}_{x:A}$ of types.

Example (cont.)

The type of dependent functions is $f : \Pi x : A.\, B$, the product of a family $\{Bx\}_{x:A}$ of types.

Intuitively $$\Pi x : A.\, Bx \;=\; \left\{ f : (A \to \bigcup_{x:A} Bx) \mid \forall a : A.\, (fa : Ba) \right\}$$

The typing rules associated are

$$(\text{abstraction}) \quad \frac{\Gamma, x : A \;\vdash\; b : B}{\Gamma \;\vdash\; \lambda x : A.b : (\Pi x : A.\, B)}$$

$$(\text{application}) \quad \frac{\Gamma \;\vdash\; f : (\Pi x : A.\, B) \quad \Gamma \;\vdash\; a : A}{\Gamma \;\vdash\; f\, a : B[x := a]}$$

Note substitution $[x := a]$ in the type of the application.

So, the formula $\forall x : X.\, Qxx \supset \bot$ is translated as the dependent function type

$$\Pi x : X.\, Qxx \to \bot$$

Example (cont.)

Therefore,

$$\begin{array}{lcl} (\mathsf{AntiSym}\, X Q) & = & \Pi x, y : X.\, Qxy \to (Qyx \to \bot) \\ (\mathsf{Irrefl}\, X Q) & = & \Pi x : X.\, Qxx \to \bot \end{array}$$

To prove that anti-symmetry implies irreflexivity for binary relations we have to find a proof-term of type

$$\Pi X : \mathsf{Set}.\, \Pi R : (\mathsf{Rel} X).\, (\mathsf{AntiSym}\, X R) \to (\mathsf{Irrefl}\, X R)$$

the following term is of this type

$$\lambda X : \mathsf{Set}.\, \lambda R : (\mathsf{Rel} X).\, \lambda h : (\mathsf{AntiSym}\, X R).\, \lambda x : X.\, \lambda q : (Rxx).\, hxxqq$$

The verification of this claim is performed by the type-checking algorithm.

Simply-typed λ-calculus is not enough

Simply-typed λ-calculus has not enough expressive power to encode the kind of logic used in the previous example.

There are several type systems embedding some of the features described in our example. For example:

- **System F** – features polymorphism
- **λP** – features dependent types
- **System Fω**– features higher-order polymorphism
- **CC** – features dependent types and higher-order polymorphism

There is a general class of typed λ-calculi were all these systems can be described – the **Pure Type Systems**.