Lecture 5: Probabilistic systems

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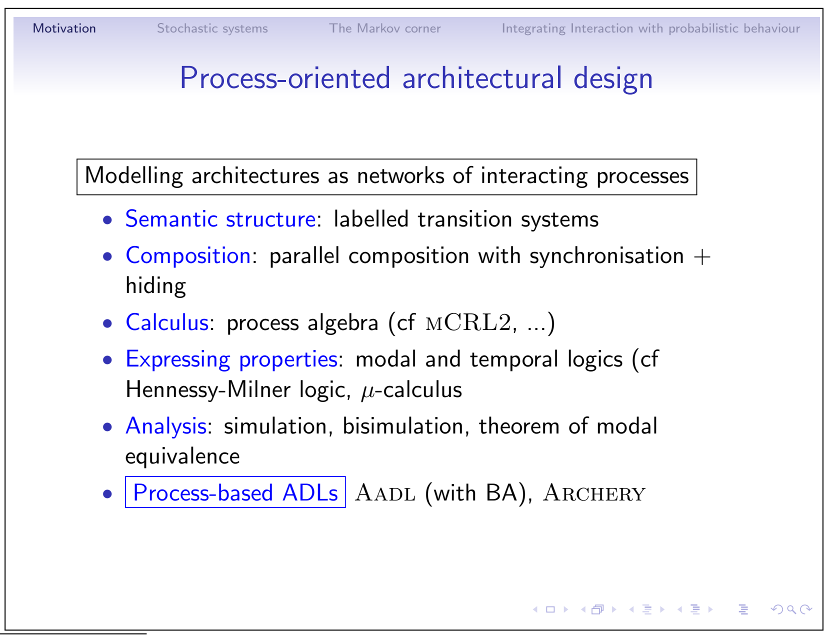
Abstract

This lecture discusses reactive systems with stochastic behaviour, modelled as interactive Markov chains.

1 Motivation



Lecture notes for *Arquitectura e Cálculo*, MEI profile in *Formal Methods in Software Engineering*, 2014-15.

... with stochastic behaviour

Specification and analysis of architectures with stochastic constrains

- combine interactive transitions and probabilistic transitions
- combine process algebra with stochastic processes
- increasing modelling and analysis power
- semantic model: Interactive Markov chains [Hermanns, 2002]
- tools
- ADLs with stochastic constraints [Aldini, 2011]

2 Stochastic systems

Random variables & Distributions

Random variable

$X : \Omega \longrightarrow S$ where S is a set of states and $\langle \Omega, F, \mathbf{P} \rangle$ is a probability space.

Probability space $\langle \Omega, F, \mathbf{P} \rangle$

- F is a σ-algebra of 'events': a family of subsets of Ω, including $\emptyset$ and Ω, closed under complement, countable unions and intersections
- $\mathbf{P} : F \longrightarrow [0, 1]$ is a probability measure satisfying the Kolmogorov axioms:
  - $\mathbf{P}[\Omega] = 1$
  - for any set of disjoint 'events' $A_1, ..., A_n \in F$

$$\mathbf{P}[A_1 \cup A_2 \cup ... \cup A_n] = \mathbf{P}[A_1] + \mathbf{P}[A_2] + ... + \mathbf{P}[A_n]$$

Often the 'state space' is not observable and X defined by its distribution

Random variables & Distributions

CFD (cumulative distribution function)

$$\begin{aligned} F_X\, s &= \mathbf{P}[\{\omega \in \Omega \mid X\,\omega \leq s\}] \\ &= \mathbf{P}[X \leq s] \end{aligned}$$

Discrete vs continuous random variable

$$F_X\, s = \sum_{k \leq s} p_X\, k \qquad \text{vs} \qquad F_X\, s = \int_{s_0}^{s} f_X\, s$$

where

- $p_X\, s = \mathbf{P}[X = s]$
  (probabilistic mass function)
- $f_X\, s = \mathbf{P}[X \in ds]$, for an infinitesimal interval ds centered around s
  (probabilistic density function)

Stochastic process

... to study random effects which change over time

Stochastic process

is a family $\{X_t : \Omega \longrightarrow S \mid t \in T\}$ of random variables over the same probability space

discrete/continuous time/space

3 The Markov corner

Markov

The Markov condition

The future behaviour is totally independent of past history

$$\mathbf{P}\left[X_{t'} \in A \,|\, X_t = P, X_{t-1} = P_{t-1}, \cdots, X_0 = P_0\right] \;=\; \mathbf{P}\left[X_{t+1} \in A \,|\, X_t = P\right]$$

Homogeneous time condition

Behaviour is totally independent of the observation instant

$$\mathbf{P}\left[X_{t+1} \in A \mid X_t = P\right] \;=\; \mathbf{P}\left[X_{t'-t} \in A \mid X_0 = P\right]$$

(for the discrete case $T \cong \boldsymbol{N}$)

Markov chains vs Markov processes

Discrete Time Markov chains

... (Discrete Time) Markov chains as transition systems

DTMC

$$\begin{aligned} & \mathbf{P}\left[X_{t'} = P' \mid X_t = P, X_{t-1} = P_{t-1}, \cdots, X_0 P_0\right] \\ = \; & = \mathbf{P}\left[X_{t+1} = P' \mid X_t = P\right] \\ = \; & = \mathbf{P}\left[X_1 = P' \mid X_t 0 = P\right] \end{aligned}$$

Probabilistic TS

$$\langle S, \longrightarrow \subseteq S \times \boldsymbol{R}^{+} \times S \rangle$$

such that for each state probabilities of outgoing transition cumulate to 1
Defines a probabilistic chain if added an initial state/distribution

Discrete Time Markov chains

Sojourn or holding time

The number of consecutive time steps the process remains in a given a state before exiting is a random variable geometrically distributed

$$\mathbf{P}[SJ_s = i] \;=\; p^i(1-p)$$

where p is the looping probability at s

- memoryless: the information that a process has been in a state for a certain amount of time is irrelevant for the distribution of the residual sojourn time (consequence of the Markov condition)
- The geometric distribution is the only memoryless discrete distribution

Continuous Time Markov chains

... (Continuous Time) Markov chains as transition systems

CTMC

For an arbitrary sequence of instants $t_{n+\Delta t} > t_n > t_{n-1} > \cdots > t_0$

$$\begin{array}{ll} & \mathbf{P}[X_{t_{n+\Delta t}} = P' \mid X_{t_n} = P, X_{t_{n-1}} = P_{t_{n-1}}, \cdots, X_{t_0} P_{t_0}] \\ = & \mathbf{P}[X_{t_{n+\Delta t}} = P' \mid X_{t_n} = P] \\ = & \mathbf{P}[X_{\Delta t} = P' \mid X_{t_0} = P] \end{array}$$

Transitions are independent of history and observation instant, but not of interval Δt:

$$\mathbf{P}[X_{\Delta t} = P' \mid X_{t_0} = P] \;=\; \lambda \Delta t + o(\Delta t)$$

Continuous Time Markov chains

Transition rate λ

- defines how the one-step transition probability between states P and P' increases with time
- domes not depend on the length of the interval

Markovian TS

$$\langle S, \longrightarrow \subseteq S \times \mathbf{R}^{+} \times S \rangle$$

Defines a Markovian chain if added an initial state/distribution

The probabilistic behaviour of a CTMC is completely described by the initial state and the transition rates between distinct states

Continuous Time Markov chains

Sojourn or holding time

... is exponentially distributed

$$\mathbf{P}\left[SJ_s \leq t\right] \;=\; 1 - e^{\lambda t}$$

- The exponential distribution is the only memoryless continuous distribution
- Analysis of CTMC

4 Integrating interaction with probabilistic behaviour

Integration

Interactive behaviour + DTMC

Interactive behaviour + CTMC

Interactive Markov chains (IMC)

Motivation Stochastic systems The Markov corner Integrating Interaction with probabilistic behaviour

Integration

- some after thoughts
- an application of IMC to `Reo`