Pre / post-conditions — starting where (pure) functions stop

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What if invariants are not met?

Back to the *mobile phone problem*, suppose that the requirements were (partly) misunderstood and that *store* was modelled simply as follows:

$$store : Call \to ListOfCalls \to ListOfCalls$$
$$store\ c\ l \triangleq\ c : l$$

Clearly, *store* **fails** to preserve invariant *ListOfCalls* in case

- $length\ l = 10$, or
- $c \in elems\ l$, equivalent to $\langle \exists\ i\ :\ 1 \leq i \leq length\ l :\ l\ i = c \rangle$

NB: $elems\ l \triangleq\ \{l\ i : i \in inds\ l\}$ yields the set of all elements of a finite list l, where $inds\ l$ denotes the set of all indices of l, that is, $inds\ [] = \{\}$ and $inds\ l = \{1, \ldots, length\ l\}$ otherwise.

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Need for pre-conditions

- So, designers would have to **restrict** the application of *store* to input values c, l such that the invariant is preserved.
- This could be achieved by adding a **pre-condition**:

$$store : Call \to ListOfCalls \to ListOfCalls$$
$$store\ c\ l \triangleq\ c : l$$
$$\textbf{pre}\ length\ l < 10 \wedge c \notin elems\ l$$

- Such a pre-condition is a predicate telling a range of **acceptable** input values — to be read as a *warning* provided by the designer that the function may misbehave outside such a range of values.

(Pure) functions are not enough

Thus

- *store* would become a **partial function** (clearly a symptom that the requirements had been misunderstood)

However,

- Partial functions are the rule (rather than the exception) in mathematics and computing.

Examples:

- Numbers — we know what $1/2$ means; what about $1/0$? — *division* is a **partial** function
- List processing: given a sequence s, what does $s\ i$ mean in case $i > \mathit{length}\ s$? — list *indexing* is a **partial** operation.

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Pre-conditions for safety

Since

- the formal **meaning** of a program always be a well-defined mathematical object ;
- one has to ensure that **no** partial function is used outside its domain of definition,

the following strategy is recommended for **safety**, in presence of partial functions:

- Write your model as if all functions were total
- Chase the partial ones and add predicates pre-conditions ensuring that all such functions are called within their domain of definition.

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Pre-conditions for safety

Example: wishing to specify the operation which subtracts the first from the second element of a finite sequence of natural numbers,

$$Sub21 : \mathbb{N}^\star \to \mathbb{N}$$
$$Sub21\ s \triangleq s\ 2 - s\ 1$$

we realize that the argument list is *required* to have at least two elements. So we add a pre-condition

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Pre-conditions for safety

However, subtraction in $\mathbb{N}$ is a partial function too. So we add another clause to the precondition:

$$\begin{aligned} &\mathit{Sub}21 : \mathbb{N}^\star \to \mathbb{N} \\ &\mathit{Sub}21\ s \triangleq\ s\ 2 - s\ 1 \\ &\textbf{pre}\ \mathit{length}\ s \geq 2\ \wedge\ s\ 2 > s\ 1 \end{aligned} \tag{18}$$

What if the specifier decides to write clause

$$\textbf{pre}\ \mathit{length}\ s = 2\ \wedge\ s\ 2 > s\ 1 \tag{19}$$

instead?

Weakest preconditions

Clearly,

- both (18) and (19) are suitable pre-conditions for *Sub*21
- (19) is **stronger** than (18), since $\mathit{length}\ l = 2 \Rightarrow \mathit{length}\ l \geq 2$
- (18) is therefore "better" than (19), as the latter restricts the use of *Sub*21 too much.

It turns out that

- predicate (18) is the **weakest pre-condition** (WP) for *Sub*21 to be safe
- one should aim at always *stopping* at WPs.

We will learn later how to **calculate** WPs. A thumb rule is given in the next slide for a special (in fact, easiest) case.

Weakest preconditions

Let $f : X \to Y$ be a function where type Y is constrained by an invariant, inv-$Y : Y \to \mathbb{B}$. Then the **weakest pre-condition** to be enforced on f with respect to inv-Y is

$$wp(f, \text{inv-}Y)\ x \quad \triangleq \quad \text{inv-}Y(f\ x) \tag{20}$$

Exercise 8: Calculate the weakest precondition $wp(f, \text{inv-}Y)$ for each situation below:

X	Y	$f\ x$	inv-Yy
$\mathbb{N}_0$	$\mathbb{N}$	$f\ x \triangleq x^2 + 1$	$y \leq 10$
$\mathbb{N}_0$	$\mathbb{N}$	the same	$1 \leq y$
$\mathbb{N}_0$	$\mathbb{N}$	$f = succ$	$even\ y$
$\mathbb{N} \times \mathbb{N}^\star$	$\mathbb{N}^\star$	$f(n, x) \triangleq n : x$	$\langle \forall\ m\ :\ m \in elems\ y :\ m \leq 10 \rangle$

□

Weakest preconditions

Exercise 9: Indicate which predicates p below are stronger (or weaker) than the weakest precondition (WP) on each f with respect to the corresponding output invariant:

X	Y	f	inv-$Y(y)$	$p(x)$
$\mathbb{R}$	$\mathbb{R}$	$f\ x \triangleq x^2 + 1$	$0 \leq y \leq 10$	$0 < x < 3$
$\mathbb{N}^\star$	$\mathbb{N}^\star$	$f = map\ \underline{1}$	$\langle \forall\ i\ :\ i \in \mathsf{inds}\, y :\ y\ i > 10 \rangle$	$\textsc{True}$
$A^\star$	$A^\star$	$f = tail$	$length\ y > 0$	$x \neq [\,]$
$BTree\ A$	$BTree\ A$	$f = mirror$	$depth\ y \geq 1$	$depth\ x > 1$

where *map* and *tail* are well known list operators and *mirror* and *depth* are the obvious functions over binary trees.

□

Need for more

When studying **probability** theory and **statistics** one is faced with problems such as the following:

> *One is picking up marbles from a bag initially with a red, a blue and a yellow marble. Compute the probability of the experiment in which red is picked first, yellow second and blue third.*

Suppose you want to build an abstract model of a program you want to run as much as possible to confirm the theory:

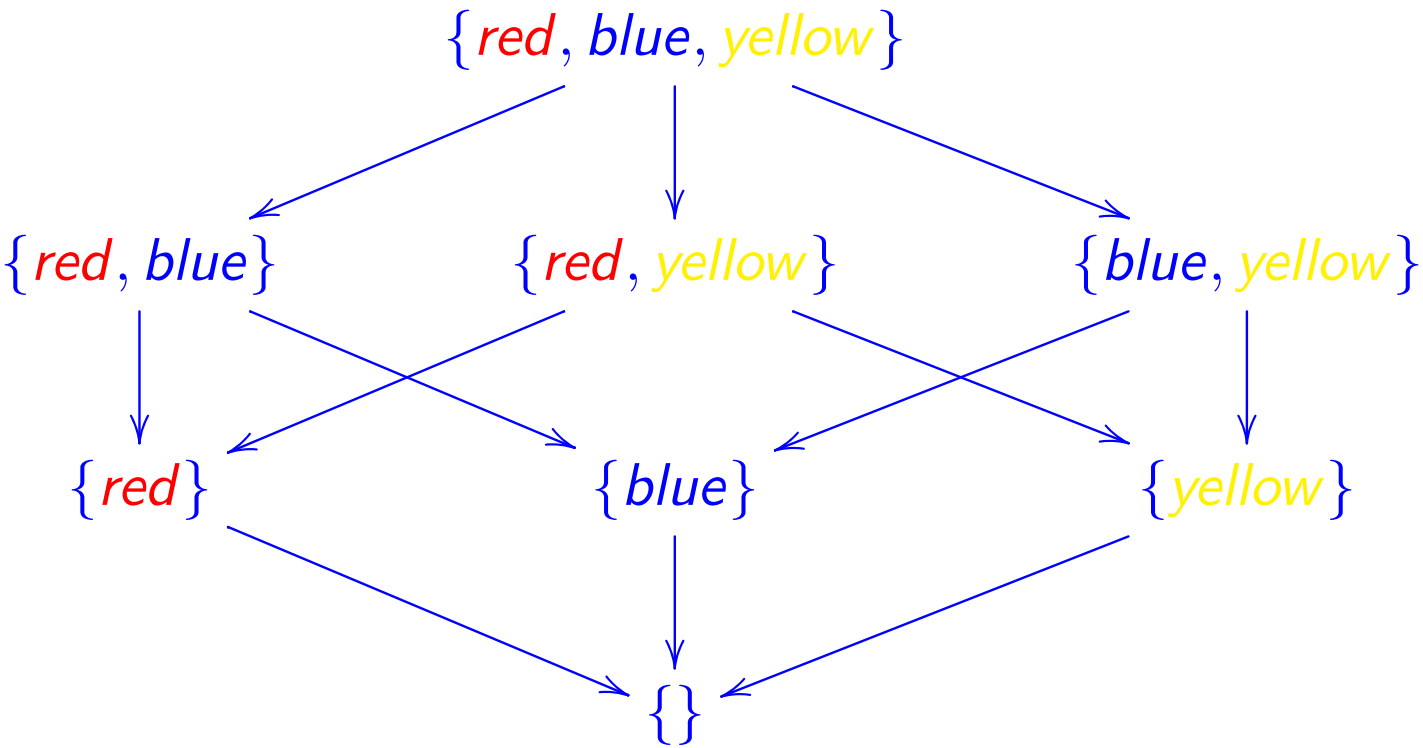
- Datatypes:

$$Marble = \{red, blue, yellow\}$$
$$Bag = \{B : B \subseteq Marble\}$$

NB: one may alternatively write $Bag = \mathcal{P}\,Marble$, see next slide.

Need for more

The extension of *Bag* is as follows:



This is known as the **powerset lattice** of set *Marble*.

Need for more

Operations: one needs the operation which puts all marbles back into the bag

$$reset : Bag \to Bag$$
$$reset\ b \triangleq \{red, blue, yellow\}$$

and another to simulate the experiment of picking the next marble:

$$Pick : Bag \to (Marble \times Bag)$$
$$Pick\ b \triangleq \ldots$$

However, for the experiment to be valid, the choice of the next marble to pick must be **non-deterministic**: *Pick* is **not** a function!

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Post-conditions for liveness

Let

- x denote a marble to be taken from bag b
- r denote b without such a marble

The best we can say about the experiment is

$$x \in b \ \wedge \ r = b - \{x\}$$

assuming $b \neq \{\}$.

We are led to a specification based on a **pre-/post**-condition pair:

$$Pick : (x : Marble, r : Bag) \leftarrow (b : Bag)$$
$$\textbf{pre } b \neq \{\}$$
$$\textbf{post } \ x \in b \ \wedge \ r = b - \{x\}$$

Post-conditions for vagueness

- Another use of **pre-/post-** pairs is that of tolerating more than one result
- Example: we want to specify *"the function"* **square root** of an integer:

$$Sqrt : (r : \mathbb{R}) \leftarrow (i : \mathbb{Z})$$
$$\textbf{pre } i \geq 0$$
$$\textbf{post } \ r^2 = i$$

The specifier is telling the implementer that either solution $r = +\sqrt{i}$ or $r = -\sqrt{i}$ will do.

Post-conditions for implicit specification

- Post-conditions — elegant way of *hiding* algorithmic details which a particular function always embodies.
- Wherever we write a post-condition bearing in mind to specify a function f, we refer to such a condition as an **implicit specification** of f.

Example: **explicit** definition of *abs* function

$$abs : \mathbb{Z} \to \mathbb{N}$$
$$abs\ i \triangleq \textbf{if } i < 0 \textbf{ then } -i \textbf{ else } i$$

followed by **implicit** definition of the same function:

$$abs : (i : \mathbb{Z}) \to (r : \mathbb{N})$$
$$\textbf{post } r \geq 0 \wedge (r = i \vee r = -i)$$

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Examples

Explicit definition of *max* function

$$\begin{aligned} &max : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z} \\ &max(i,j) \triangleq \quad \textbf{if } i \leq j \textbf{ then } j \textbf{ else } i \end{aligned} \tag{21}$$

followed by its **implicit** specification:

$$\begin{aligned} &max : (i : \mathbb{Z}, j : \mathbb{Z}) \to (r : \mathbb{Z}) \\ &\textbf{post} \quad r \in \{i,j\} \wedge i \leq r \wedge j \leq r \end{aligned} \tag{22}$$

Now the implicit specification of a partial function:

$$\begin{aligned} &Maxs : (s : \mathcal{P}\mathbb{N}) \to (r : \mathbb{N}) \\ &\textbf{pre } s \neq \{\} \\ &\textbf{post} \quad r \in s \wedge \langle \forall\ i \ :\ i \in s :\ i \leq r \rangle \end{aligned}$$

A glimpse at deriving **explicit** from **implicit**

The "best" specification of *max* is as follows, cf. its post-condition:

$$max(i,j) \leq r \;\equiv\; i \leq r \wedge j \leq r \tag{23}$$

Let us calculate *explicit* definition (21) from (23):

- Case $i \leq j = \text{TRUE}$:

$$\begin{array}{ll} & max(i,j) \leq r \;\equiv\; i \leq r \wedge j \leq r \\ = & \quad \{\; i \leq r \Leftarrow i \leq j \wedge j \leq r \;\} \\ & max(i,j) \leq r \;\equiv\; j \leq r \\ :: & \quad \{\; \text{indirect equality (more about this later on...)} \;\} \\ & max(i,j) \;=\; j \end{array}$$

A glimpse at deriving **explicit** from **implicit**

- Case $j \leq i = \textsc{True}$:

$$
\begin{array}{rl}
 & max(i,j) \leq r \;\equiv\; i \leq r \wedge j \leq r \\
= & \quad \{\; j \leq r \Leftarrow j < i \wedge i \leq r \;\} \\
 & max(i,j) \leq r \;\equiv\; i \leq r \\
:: & \quad \{\; \text{indirect equality (more about this later on...)} \;\} \\
 & max(i,j) \;=\; i
\end{array}
$$

Putting both cases together:

$$max(i,j) \triangleq \textbf{ if } i \leq j \textbf{ then } j \textbf{ else } i$$

Post-conditions for relational specification

We want to specify the **prefix** relation between two finite sequences, eg.

$$\begin{array}{rcl} [1,2] & \mathit{IsPrefixOf} & [1,2,4,1] \\ [] & \mathit{IsPrefixOf} & [] \\ [] & \mathit{IsPrefixOf} & [1] \end{array}$$

We write:

$$\mathit{IsPrefixOf} : (s : A^\star) \leftarrow (r : A^\star)$$
$$\textbf{post} \;\; \mathit{length}\ s \leq \mathit{length}\ r \wedge \langle \forall\ i\ :\ i \leq \mathit{length}\ s :\ (s\ i) = (r\ i) \rangle$$

NB: note that this spec is parametric on A.

Post-conditions for relational specification

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NB: note that this spec is parametric on A.

Post-conditions for relational specification

Another example: the relation which expresses sequence permutation:

$$\begin{aligned} &\mathit{Permutes} : (s : A^\star) \leftarrow (r : A^\star) \\ &\textbf{post} \quad \langle \forall\ a\ :\ a \in \mathit{elems}(s +\!\!+ r) :\ \mathit{count}\ a\ s = \mathit{count}\ a\ r \rangle \end{aligned} \tag{24}$$

assuming

$$\begin{aligned} &\mathit{count} : A \to A^\star \to \mathbb{N}_0 \\ &\mathit{count}\ a\ s \triangleq\ \mathit{card}\{i : i \in \mathit{inds}\ s \wedge (s\ i) = a\} \end{aligned}$$

where $\mathit{card} : \mathcal{P}A \to \mathbb{N}_0$ computes the number of elements of a finite set.

Example: sorting

The following implicit specification of **sorting** abstracts from the particular algorithm one has in mind:

$$\begin{aligned} & \mathit{Sort} : (s : A^\star) \leftarrow (r : A^\star) \\ & \textbf{post} \quad \mathit{isOrdered}(\leq)\ s \wedge s\ \mathit{Permutes}\ r \end{aligned} \tag{25}$$

As seen in the following exercise, predicate *isOrdered* assumes a total order $(\leq)$ on datatype A.

Exercise 10: Complete the following (inductive) specification of *isOrdered*:

$$\begin{aligned} & \mathit{isOrdered}(\leq)[\,] = \textsc{True} \\ & \mathit{isOrdered}(\leq)(a : x) = \ldots \mathit{isOrdered}(\leq)x \ldots \end{aligned}$$

□

Exercises

Exercise 11: Give an implicit definition for function $f\ x \triangleq x^2 + 1$ over the natural numbers.

□

Exercise 12: A *golden multiple* of a given length is obtained by multiplying this length by a real number whose square equals its "successor". Write an implicit specification for *golden multiple*.

□

Exercise 13: Write implicit and explicit specifications for function $inseq : \mathbb{N}_0 \to \mathbb{N}^\star$ which, for argument n, yields the sequence $[1, \ldots, n]$.

□

The **pre/post/inv** trilogy

By writing

$$Spec : (b : B) \leftarrow (a : A)$$
pre ...
post ...

we mean the definition of two predicates

$$\text{pre-}Spec : A \to \mathbb{B}$$
$$\text{post-}Spec : B \times A \to \mathbb{B}$$

such that

$$\langle \forall\ a\ :\ a \in A :\ \text{pre-}Spec\ a \Rightarrow \langle \exists\ b\ :\ b \in B :\ \text{post-}Spec(b, a) \rangle \rangle \quad (26)$$

Proof obligation: satisfiability

Thus (26) is another proof obligation known as **satisfiability**:

> **Satisfiability** *ensures that pre-Spec and post-Spec are such that, for all acceptable inputs, there must be some possible result.*

This includes the situation in which A and B have invariants.

Exercise 14: Assuming that the implicit definition of a total function $B \xleftarrow{f} A$ uniquely determines f, that is

$$\text{post-}f(r, a) \quad \equiv \quad r = f\ a \tag{27}$$

holds, use the Eindhoven quantifier calculus to show that (26) reduces to $\langle \forall\ a\ :\ a \in A :\ (f\ a) \in B \rangle$ for $Spec := f$. In summary: in the case of functions, *satisfiability* is the same as *invariant preservation.*

□

Exercises

Exercise 15: Consider datatype

$$NRSeq\ A = A^\star$$
$$\textbf{inv}\ \ x \triangleq\ length\ x = card(elems\ x)$$

1. What is the informal meaning of the type's invariant?
2. Tell which of the following new types for *Permutes* (24),

$$Permutes : (s : NRSeq\ A) \leftarrow (r : A^\star) \tag{28}$$
$$Permutes : (s : NRSeq\ A) \leftarrow (r : NRSeq\ A) \tag{29}$$

would lead to a non satisfiable specification.

□

Exercises

Exercise 16: Back to

$$Permutes : (s : A^\star) \leftarrow (r : A^\star)$$
$$\textbf{post} \quad \langle \forall\ a\ :\ a \in elems(s +\!\!+ r) :\ count\ a\ s = count\ a\ r \rangle$$

show that

1. *Permutes* is a **reflexive** relation: $x\ Permutes\ x \equiv \textsc{True}$ for all x.
2. *Permutes* is a **symmetric** relation: $y\ Permutes\ x \equiv x\ Permutes\ y$ for all x, y.

□

Exercise 17: How would you write an explicit definition of (partial) function *Maxs*?

□

Background — Eindhoven quantifier calculus (cont.)

Splitting:

$$\langle \forall\ k\ :\ R \vee S :\ T \rangle \quad = \quad \langle \forall\ k\ :\ R :\ T \rangle \wedge \langle \forall\ k\ :\ S :\ T \rangle \tag{30}$$

$$\langle \exists\ k\ :\ R \vee S :\ T \rangle \quad = \quad \langle \exists\ k\ :\ R :\ T \rangle \vee \langle \exists\ k\ :\ S :\ T \rangle \tag{31}$$

Rearranging:

$$\langle \forall\ k\ :\ R :\ T \wedge S \rangle \quad = \quad \langle \forall\ k\ :\ R :\ T \rangle \wedge \langle \forall\ k\ :\ R :\ S \rangle \tag{32}$$

$$\langle \exists\ k\ :\ R :\ T \vee S \rangle \quad = \quad \langle \exists\ k\ :\ R :\ T \rangle \vee \langle \exists\ k\ :\ R :\ S \rangle \tag{33}$$

de Morgan:

$$\neg\langle \forall\ k\ :\ R :\ T \rangle \quad = \quad \langle \exists\ k\ :\ R :\ \neg T \rangle \tag{34}$$

$$\neg\langle \exists\ k\ :\ R :\ T \rangle \quad = \quad \langle \forall\ k\ :\ R :\ \neg T \rangle \tag{35}$$