PF transform: conditions, coreflexives and design by contractand

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Recall

Some basic rules of the PF-transform:

ϕ	$PF\ \phi$
$\langle \exists\ a\ ::\ b\ R\ a \wedge a\ S\ c \rangle$	$b(R \cdot S)c$
$\langle \forall\ a, b\ ::\ b\ R\ a \Rightarrow b\ S\ a \rangle$	$R \subseteq S$
$\langle \forall\ a\ ::\ a\ R\ a \rangle$	$id \subseteq R$
$b\ R\ a \wedge c\ S\ a$	$(b, c)\langle R, S \rangle a$
$b\ R\ a \wedge d\ S\ c$	$(b, d)(R \times S)(a, c)$
$b\ R\ a \wedge b\ S\ a$	$b\ (R \cap S)\ a$
$b\ R\ a \vee b\ S\ a$	$b\ (R \cup S)\ a$
$(f\ \ b)\ R\ (g\ \ a)$	$b(f^{\circ} \cdot R \cdot g)a$
TRUE	$b\ \top\ a$
FALSE	$b\ \bot\ a$

Question

- The PF-transform seems applicable to transforming **binary** predicates only, easily converted to binary relations, eg. $\phi(y,x) \triangleq\ y-1=2x$ which transforms to function $y=2x+1$, etc.

- What about transforming predicates such as the following

$$\langle \forall\ x,y\ \ :\ \ y=2x \wedge \mathit{even}\ x :\ \ \mathit{even}\ y \rangle \tag{106}$$

expressing the fact that function $y=2x$ preserves even numbers, where $\mathit{even}\ x \triangleq\ \mathit{rem}(x,2)=0$ is a **unary** predicate?

Observation

- As already noted, (106) is a proposition stating that function $y = 2x$ *preserves* even numbers.
- In general, a function $A \xleftarrow{f} A$ is said to **preserve** a given predicate ϕ iff the following holds:

$$\langle \forall\ x\ :\ \phi\ x :\ \phi\ (f\ x) \rangle \tag{107}$$

- Proposition (107) itself is a particular case of

$$\langle \forall\ x\ :\ \phi\ x :\ \psi\ (f\ x) \rangle \tag{108}$$

which states that f **ensures** property ψ on its output every time property ϕ holds on its input.

Answer

We first PF-transform the scope of the quantification:

$$
\begin{array}{ll}
 & y = 2x \wedge even\ x \\
\equiv & \{ \text{ introduce } z \text{ (}\exists\text{-one-point) } \} \\
 & \langle \exists\ z\ :\ z = x :\ y = 2z \wedge even\ z \rangle \\
\equiv & \{ \ \exists\text{-trading ; introduce } \Phi_{even} \ \} \\
 & \langle \exists\ z\ ::\ y = 2z \wedge \underbrace{z = x \wedge even\ z}_{z \Phi_{even} x} \rangle \\
\equiv & \{ \text{ composition ; introduce } twice\ z \triangleq 2z \ \} \\
 & y(twice \cdot \Phi_{even})x
\end{array}
$$

cf. diagram

$$
\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "twice"'] & \mathbb{N}_0 \arrow[l, "\Phi_{even}"'] \\
\mathbb{N}_0
\end{tikzcd}
$$

Now the whole thing

$$\langle \forall\ x, y\ :\ y = 2x \wedge even\ x :\ even\ y \rangle$$
$$\equiv \quad \{\ \text{above}\ \}$$
$$\langle \forall\ x, y\ :\ y(twice \cdot \Phi_{even})x :\ even\ y \rangle$$
$$\equiv \quad \{\ \exists\text{-one-point}\ \}$$
$$\langle \forall\ x, y\ :\ y(twice \cdot \Phi_{even})x :\ \langle \exists\ z\ :\ z = y :\ even\ z \rangle \rangle$$
$$\equiv \quad \{\ \text{predicate calculus: } p \wedge \textsc{True} = p\ \}$$
$$\langle \forall\ x, y\ :\ y(twice \cdot \Phi_{even})x :\ \langle \exists\ z\ ::\ z = y \wedge even\ z \wedge \textsc{True} \rangle \rangle$$
$$\equiv \quad \{\ \top \text{ is the top relation}\ \}$$
$$\langle \forall\ x, y\ :\ y(twice \cdot \Phi_{even})x :\ \langle \exists\ z\ ::\ y \Phi_{even} z \wedge z \top x \rangle \rangle$$
$$\equiv \quad \{\ \text{composition}\ \}$$

Now the whole thing

$$
\begin{array}{ll}
& \langle \forall\, x, y \;:\; y(twice \cdot \Phi_{even})x : \; y(\Phi_{even} \cdot \top)x \rangle \\
\equiv & \qquad \{ \text{ go pointfree (inclusion) } \} \\
& twice \cdot \Phi_{even} \;\subseteq\; \Phi_{even} \cdot \top
\end{array}
$$

cf. diagram

$$
\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "twice"'] \arrow[dr, phantom, "\subseteq"] & \mathbb{N}_0 \arrow[l, "\Phi_{even}"'] \arrow[d, "\top"] \\
\mathbb{N}_0 & \mathbb{N}_0 \arrow[l, "\Phi_{even}"]
\end{tikzcd}
$$

In summary

In the calculation above, **unary** predicate *even* has been PF-transformed in two ways:

- Φ_{even} such that

$$z\ \Phi_{even} x \quad \triangleq \quad z = x \wedge even\ z$$

Clearly, $\Phi_{even} \subseteq id$ — that is, Φ_{even} is a **coreflexive** relation;

- $\Phi_{even} \cdot \top$, which is such that

$$z(\Phi_{even} \cdot \top)x \ \equiv \ even\ z$$

— a so-called (left) **condition**.

Coreflexives

As *id* can be represented as the "all-**1**s" diagonal matrix, so do **coreflexives**, which are *sub-diagonal* matrices, eg.

$$\Phi_{vowel} \quad =$$

	a	b	c	d	e	f	...
a	1	0	0	0	0	0	0
b	0	0	0	0	0	0	0
c	0	0	0	0	0	0	0
d	0	0	0	0	0	0	0
e	0	0	0	0	1	0	0
f	0	0	0	0	0	0	0
...	0	0	0	0	0	0	...

where *vowel* is the predicate identifying characters which are vowels.

Coreflexives

PF-transform of **unary** predicate p into the corresponding fragment Φ_p of *id* (coreflexive),

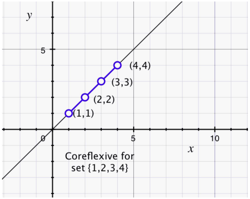
$$y\ \Phi_p\ x \ \equiv\ y = x \wedge p\ y \tag{109}$$

is unique — thus the universal property:

$$\Phi = \Phi_p \ \equiv\ (y\ \Phi\ x \equiv y = x \wedge p\ y) \tag{110}$$

A set S can also be PF-transformed into a coreflexive by calculating $\Phi_{(\in S)}$, cf. eg. the transform of set $\{1, 2, 3, 4\}$:

$$\Phi_{1 \leq x \leq 4} \quad =$$



Exercises

Exercise 51: Let false be the "everywhere false" predicate such that $\mathit{false}\ x = \text{FALSE}$ for all x, that is, $\mathit{false} = \underline{\text{FALSE}}$. Show that $\Phi_{\mathit{false}} = \bot$.

$\square$

Exercise 52: Given a set S, let Φ_S abbreviate coreflexive $\Phi_{(\in S)}$. Use (109) to unfold $\Phi_{\{1,2\}} \cdot \Phi_{\{2,3\}}$ to pointwise notation.

$\square$

Exercise 53: Show that (110) follows from (109).

$\square$

Exercise 54: Solve (110) for p under substitution $\Phi := \mathit{id}$.

$\square$

Boolean algebra of coreflexives

Building up on the exercises above, from (110) one easily draws:

$$\Phi_{p \wedge q} = \Phi_p \cdot \Phi_q \tag{111}$$

$$\Phi_{p \vee q} = \Phi_p \cup \Phi_q \tag{112}$$

$$\Phi_{\neg p} = id - \Phi_p \tag{113}$$

$$\Phi_{false} = \bot \tag{114}$$

$$\Phi_{true} = id \tag{115}$$

where p, q are predicates.

(Note the slight, obvious abuse in notation.)

Basic properties of coreflexives

Let Φ, Ψ be coreflexive relations. Then the following properties hold:

- Coreflexives are **symmetric** and **transitive**:

$$\Phi^\circ = \Phi = \Phi \cdot \Phi \tag{116}$$

- **Meet** of two coreflexives is composition:

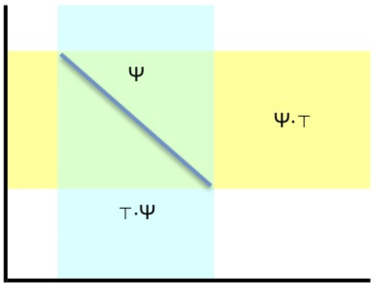
$$\Phi \cap \Psi = \Phi \cdot \Psi \tag{117}$$

- Closure properties:

$$R \cdot \Phi \subseteq S \quad \equiv \quad R \cdot \Phi \subseteq S \cdot \Phi \tag{118}$$

$$\Phi \cdot R \subseteq S \quad \equiv \quad \Phi \cdot R \subseteq \Phi \cdot S \tag{119}$$

Relating coreflexives with conditions



Coreflexive Ψ as a right-condition

$$\top \cdot \Psi$$

or as a left-condition:

$$\Psi \cdot \top$$

Mapping back and forward:

$$\Phi \subseteq \Psi \quad \equiv \quad \Phi \subseteq \top \cdot \Psi \qquad (120)$$

$$\Phi \subseteq \Psi \quad \equiv \quad \Phi \subseteq \Psi \cdot \top \qquad (121)$$

Relating coreflexives with conditions

Pre and post restriction:

$$
\begin{aligned}
R \cdot \Phi &= R \cap \top \cdot \Phi && (122)\\
\Psi \cdot R &= R \cap \Psi \cdot \top && (123)
\end{aligned}
$$

Putting these together we obtain **selection**, as in SQL:

$$\sigma_{\Psi,\Phi} R \;\triangleq\; \Psi \cdot R \cdot \Phi \qquad (124)$$

$$
\begin{tikzcd}
B \arrow[d, "\Psi"'] & A \arrow[l, "R"'] \arrow[d, "\Phi"] \\
B & A \arrow[l, "\sigma_{\Psi,\Phi} R"]
\end{tikzcd}
$$

Clearly,

$$\sigma_{\Psi,\Phi} R = \{(b, a) : b\ R\ a \wedge \psi\ b \wedge \phi\ a\} \qquad (125)$$

for $\Psi = \Phi_\psi$ and $\Phi = \Phi_\phi$.

Selection

Let us check (125):

$$
\begin{array}{ll}
& \sigma_{\Psi,\Phi} R \\
= & \quad \{ \text{ set theoretical meaning of a relation } \} \\
& \{(b,a) : b(\sigma_{\Psi,\Phi} R)a\} \\
= & \quad \{ \text{ definition (124) } \} \\
& \{(b,a) : b(\Psi \cdot R \cdot \Phi)a\} \\
= & \quad \{ \text{ composition } \} \\
& \{(b,a) : \langle \exists\ c\ :\ b\ \Psi\ c :\ c(R \cdot \Phi)a \rangle\} \\
= & \quad \{ \text{ coreflexive } \Psi = \Phi_\psi \text{ (110) ; } \exists\text{-trading } \} \\
& \{(b,a) : \langle \exists\ c\ :\ b = c :\ \psi b \wedge c(R \cdot \Phi)a \rangle\} \\
= & \quad \{ \text{ next slide } \}
\end{array}
$$

Selection

$$
\begin{array}{ll}
= & \{\ \exists\text{-one-point} \ ;\ \text{composition again}\ \} \\
& \{(b,a) : \psi\ b \wedge \langle \exists\ d\ ::\ b\ R\ d \wedge d\ \Phi\ a\rangle\} \\
= & \{\ \text{coreflexive}\ \Phi = \Phi_\phi\ (110)\ ;\ \exists\text{-trading}\ \} \\
& \{(b,a) : \psi\ b \wedge \langle \exists\ d\ :\ d = a :\ b\ R\ d \wedge \phi\ a\rangle\} \\
= & \{\ \exists\text{-one-point} \ ;\ \text{trivia}\ \} \\
& \{(b,a) : \psi\ b \wedge b\ R\ a \wedge \phi\ a\}
\end{array}
$$

Exercise 55: Combinator

$$R \,\square\, S \quad \triangleq \quad R \cdot \top \cdot S \tag{126}$$

is known as the "rectangular" combinator. Recalling that $\ker ! = \top$, show that $! \,\square\, !^\circ \;=\; id$

$\square$

Projection

By the way, another SQL-like relational operator is **projection**,

$$\pi_{g,f} R \;\triangleq\; g \cdot R \cdot f^{\circ} \qquad \begin{tikzcd} B \arrow[d, "g"'] & A \arrow[l, "R"'] \arrow[d, "f"] \\ C & D \arrow[l, "\pi_{g,f} R"] \end{tikzcd} \tag{127}$$

whose set-theoretic meaning is

$$\pi_{g,f} R \;=\; \{(g\ b, f\ a) : b\ R\ a\} \tag{128}$$

Functions f and g are often referred to as **attributes** of R.

Exercise 56: Check (128).

□

Exercise

Exercise 57: A relation R is said to satisfy **functional dependency** (FD) $g \rightarrow f$, written $g \xrightarrow{R} f$ wherever projection $\pi_{f,g} R$ (127) is simple.

1. Show that

$$g \xrightarrow{R} f \quad \equiv \quad \ker(g \cdot R^{\circ}) \subseteq \ker f \tag{129}$$

2. Show that (129) trivially holds wherever g is injective and R is simple, for all (suitably typed) f.

3. Prove the **composition rule** of FDs:

$$h \xleftarrow{S \cdot R} g \quad \Leftarrow \quad h \xleftarrow{S} f \quad \wedge \quad f \xleftarrow{R} g \tag{130}$$

□

Two useful coreflexives

Domain:

$$\delta\, R \quad \triangleq \quad \ker R \cap id \tag{131}$$

Range:

$$\rho\, R \quad \triangleq \quad \mathrm{img}\, R \cap id \tag{132}$$

Universal properties:

$$\delta\, R \subseteq \Phi \quad \equiv \quad R \subseteq \top \cdot \Phi \tag{133}$$

$$\rho\, R \subseteq \Phi \quad \equiv \quad R \subseteq \Phi \cdot \top \tag{134}$$

Domain/range elimination rules:

$$\top \cdot \delta\, R \quad = \quad \top \cdot R \tag{135}$$

$$\rho\, R \cdot \top \quad = \quad R \cdot \top \tag{136}$$

$$\delta\, R \subseteq \delta\, S \quad \equiv \quad R \subseteq \top \cdot S \tag{137}$$

Two useful coreflexives

More facts about domain and range:

$$\delta R = \rho (R^\circ) \tag{138}$$
$$\delta (R \cdot S) = \delta (\delta R \cdot S) \tag{139}$$
$$\rho (R \cdot S) = \rho (R \cdot \rho S) \tag{140}$$
$$R = R \cdot (\delta R) \tag{141}$$
$$R = (\rho R) \cdot R \tag{142}$$

Exercise 58: Recalling (122), (123) and other properties of relation algebra, show that: (a) (133) and (134) can be re-written with R replacing $\top$; (b) $\Phi \subseteq \Psi \equiv ! \cdot \Phi \subseteq ! \cdot \Psi$ holds.

□

Exercise

Exercise 59: Recall diagram (102) of a library loan data model:

$$
\begin{tikzcd}
ISBN \arrow[d, "M"'] & & ISBN \times UID \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] \arrow[d, "R"] \arrow[dll, phantom, "\supseteq"] \arrow[drr, phantom, "\subseteq"] & & UID \arrow[d, "N"] \\
Title \times Publisher \arrow[rr, "\top"'] & & Date & & Name \times Address \times Phone \arrow[ll, "\top"]
\end{tikzcd}
$$

Show that the invariants captured by the two rectangles can be alternatively expressed by

$$\delta\,(\pi_{id,\pi_1} R) \subseteq \delta\, M \quad \wedge \quad \delta\,(\pi_{id,\pi_2} R) \subseteq \delta\, N$$

clearly exhibiting the **foreign/primary**-key relationships of the data model (*ISBN* and *UID*).

□

Coreflexives at work — data flow

Coreflexives are very handy in controlling information flow in PF-expressions, as the following two PF-transform rules show, given two suitably typed coreflexives $\Phi = \Phi_\phi$ and $\Psi = \Phi_\psi$:

- Guarded **composition**: for all b, c

$$\langle \exists\ a\ :\ \phi\ a :\ b\ R\ a \wedge a\ Sc \rangle \quad \equiv \quad b(R \cdot \Phi \cdot S)c \qquad (143)$$

- Guarded **inclusion**:

$$\begin{array}{l} \langle \forall\ b, a\ :\ \phi\ b \wedge \psi\ a :\ b\ R\ a \Rightarrow b\ S\ a \rangle \\ \qquad \equiv\ \Phi \cdot R \cdot \Psi \subseteq S \end{array} \qquad (144)$$

For $\Phi = id$ and $\Psi = id$ we recover the (non-guarded) standard definitions.

Coreflexives at work — satisfiability

Back to the **pre**/**post** specification style, by writing

$$\begin{array}{l} Spec : (b : B) \leftarrow (a : A) \\ \textbf{pre} \;\; \ldots \\ \textbf{post} \;\; \ldots \end{array}$$

we mean the definition of two predicates

$$\begin{array}{l} \text{pre-}Spec : A \rightarrow \mathbb{B} \\ \text{post-}Spec : B \times A \rightarrow \mathbb{B} \end{array}$$

such that the **satisfiability** condition holds:

$$\langle \forall\ a\ :\ a \in A \wedge \text{pre-}Spec\ a :\ \langle \exists\ b\ :\ b \in B :\ \text{post-}Spec(b, a) \rangle \rangle \quad (145)$$

Coreflexives at work — satisfiability

Let us abbreviate

- $\Phi_{\text{pre-}Spec}$ by Pre
- $\Phi_{\text{post-}Spec}$ by $Post$
- $\Phi_{(\in A)}$ by Φ_A, which in general includes an invariant associated to datatype A
- $\Phi_{(\in B)}$ by Φ_B, which in general includes an invariant associated to datatype B

Then (145) PF-transforms to

$$
\begin{tikzcd}
A \arrow[d, "Pre"'] & & A \arrow[ll, "\Phi_A"'] \arrow[d, "Post"] \\
A & B \arrow[l, "\top"] & B \arrow[l, "\Phi_B"]
\end{tikzcd}
\qquad
Pre \cdot \Phi_A \subseteq \top \cdot \Phi_B \cdot Post \tag{146}
$$

Functional satisfiability

Case $Pre = id$, $Post = f$:

$$
\begin{array}{cl}
& \Phi_A \subseteq \top \cdot \Phi_B \cdot f \\
\equiv & \qquad \{ \text{ shunting rule (55) } \} \\
& \Phi_A \cdot f^\circ \subseteq \top \cdot \Phi_B \\
\equiv & \qquad \{ \text{ converses } \} \\
& f \cdot \Phi_A \subseteq \Phi_B \cdot \top \\
\equiv & \qquad \{ \text{ (64), since } f \cdot \Phi_A \subseteq f \ \} \\
& f \cdot \Phi_A \subseteq f \cap \Phi_B \cdot \top \\
\equiv & \qquad \{ \text{ (123) } \} \\
& f \cdot \Phi_A \subseteq \Phi_B \cdot f
\end{array}
$$

What does this mean?

Functional satisfiability ≡ invariant preservation

Let us introduce variables in $f \cdot \Phi_A \subseteq \Phi_B \cdot f$:

$$
\begin{array}{cl}
 & f \cdot \Phi_A \subseteq \Phi_B \cdot f \\
\equiv & \qquad \{ \text{ shunting rule (54) } \} \\
 & \Phi_A \subseteq f^\circ \cdot \Phi_B \cdot f \\
\equiv & \qquad \{ \text{ introduce variables } \} \\
 & \langle \forall\ a, a' \ :\ a\ \Phi_A\ a' :\ (f\ a)\Phi_B(f\ a') \rangle \\
\equiv & \qquad \{ \text{ coreflexives } (a = a') \ \} \\
 & \langle \forall\ a \ ::\ a\ \Phi_A\ a \Rightarrow (f\ a)\Phi_B(f\ a) \rangle \\
\equiv & \qquad \{ \text{ meaning of } \Phi_A,\ \Phi_B \ \} \\
 & \langle \forall\ a \ :\ a \in A :\ (f\ a) \in B \rangle
\end{array}
$$

Functional satisfiability $\equiv$ invariant preservation

Another way to put it:

$$
\begin{array}{ll}
 & f \cdot \Phi_A \subseteq \Phi_B \cdot f \\
\equiv & \quad \{ \text{ shunting } \} \\
 & f \cdot \Phi_A \cdot f^\circ \subseteq \Phi_B \\
\equiv & \quad \{ \text{ coreflexives } \} \\
 & f \cdot \Phi_A \cdot \Phi_A^\circ \cdot f^\circ \subseteq \Phi_B \\
\equiv & \quad \{ \text{ image definition } \} \\
 & \mathrm{img}\,(f \cdot \Phi_A) \subseteq \Phi_B \\
\equiv & \quad \{ \ f \cdot \Phi_A \text{ is simple } \} \\
 & \rho\,(f \cdot \Phi_A) \subseteq \Phi_B
\end{array}
$$

Functional satisfiability ≡ invariant preservation

We will write "type declaration"

$$\Phi_B \xleftarrow{f} \Phi_A \tag{147}$$

to mean

$$f \cdot \Phi_A \subseteq \Phi_B \cdot f \quad \text{cf. diagram} \quad \begin{tikzcd} A \arrow[d, "f"'] & A \arrow[l, "\Phi_A"'] \arrow[d, "f"] \\ B & B \arrow[l, "\Phi_B"] \end{tikzcd} \tag{148}$$

equivalent to both

$$f \cdot \Phi_A \subseteq \Phi_B \cdot \top \tag{149}$$

$$\rho\,(f \cdot \Phi_A) \subseteq \Phi_B \tag{150}$$

Design by contract

In general, a "type declaration" $\Psi \xleftarrow{f} \Phi$ (147) is the basis of **functional programming** (f) with so-called **contracts** (Ψ, Φ), an instance of the well-known *Design by Contract (***DbC***)* methodology (more about this later).

DbC works because **contracts** are compositional,

$$\Psi \xleftarrow{f \cdot g} \Phi \quad \Leftarrow \quad \Psi \xleftarrow{f} \Upsilon \ \wedge \ \Upsilon \xleftarrow{g} \Phi \qquad (151)$$

that is, diagram

$$\begin{tikzcd} \Psi & \Upsilon \arrow[l, "f"'] & \Phi \arrow[l, "g"'] \arrow[ll, bend left, "f \cdot g"] \end{tikzcd}$$

makes sense.

Design by contract

Contract composition (151) is easy to prove:

$$
\begin{array}{cl}
 & \Psi \xleftarrow{\;f\;} \Upsilon \;\wedge\; \Upsilon \xleftarrow{\;g\;} \Phi \\
\equiv & \qquad \{ \;\; (147)\text{ twice } \;\; \} \\
 & f \cdot \Upsilon \subseteq \Psi \cdot f \;\;\wedge\;\; g \cdot \Phi \subseteq \Upsilon \cdot g \\
\Rightarrow & \qquad \{ \;\; \text{monotonicity of } (\cdot g) \text{ and } (f\cdot) \;\; \} \\
 & f \cdot \Upsilon \cdot g \subseteq \Psi \cdot f \cdot g \;\;\wedge\;\; f \cdot g \cdot \Phi \subseteq f \cdot \Upsilon \cdot g \\
\Rightarrow & \qquad \{ \;\; \subseteq \text{ is transitive } \;\; \} \\
 & f \cdot g \cdot \Phi \subseteq \Psi \cdot f \cdot g \\
\equiv & \qquad \{ \;\; (147) \;\; \} \\
 & \Psi \xleftarrow{\;f \cdot g\;} \Phi
\end{array}
$$

Design by contract

Contracts cam also be paired, leading to the type rule (153) which is derived in the exercise below.

Exercise 60: Rely on the **absorption** property

$$\langle R \cdot T, S \cdot U \rangle \quad = \quad (R \times S) \cdot \langle T, U \rangle \tag{152}$$

in showing that

$$\Psi \times \Upsilon \xleftarrow{\langle f,g \rangle} \Phi \qquad \equiv \qquad \Psi \xleftarrow{f} \Phi \ \wedge \ \Upsilon \xleftarrow{g} \Phi \tag{153}$$

holds.

□

Exercises

Exercise 61: From (147) and properties (54), etc infer the following **DbC** rules

$$\Upsilon \xleftarrow{f} \Phi \cup \Psi \quad \equiv \quad \Upsilon \xleftarrow{f} \Phi \ \wedge \ \Upsilon \xleftarrow{f} \Psi \tag{154}$$

$$\Phi \cdot \Psi \xleftarrow{f} \Upsilon \quad \equiv \quad \Phi \xleftarrow{f} \Upsilon \ \wedge \ \Psi \xleftarrow{f} \Upsilon \tag{155}$$

You will also need $(R\cdot)$-distribution (73).

□

Exercise 62: Show that (146) means the same as

$$Pre \cdot \Phi_A \subseteq Post^\circ \cdot \Phi_B \cdot Post \tag{156}$$

□

Exercises

Exercise 63: Consider the relational version of McCarthy's conditional combinator which follows:

$$p \rightarrow f, g \;=\; f \cdot \Phi_p \cup g \cdot \Phi_{\neg p} \tag{157}$$

(a) Using (149) infer the following **DbC** rule for *conditionals*:

$$\Upsilon \xleftarrow{p \rightarrow f, g} \Psi \quad \equiv \quad \Upsilon \xleftarrow{f} \Psi \cdot \Phi_p \;\wedge\; \Upsilon \xleftarrow{g} \Psi \cdot \Phi_{\neg p} \tag{158}$$

(b) Now try and define a rule for handling contracts involving conditional conditions:

$$\Upsilon \xleftarrow{p \rightarrow f, g} (p \rightarrow \Psi, \Phi) \;=\; \ldots. \tag{159}$$

□

Exercises

Exercise 64: Recall that our motivating ESC assertion (106) was stated but not proved. Now that we know that (106) PF-transforms to

$\Phi_{even} \xleftarrow{twice} \Phi_{even}$ and that $\Phi_{even} = \rho\ twice$, complete the following *"almost no work at all"* PF-calculation of (106):

$$
\begin{array}{cl}
 & \Phi_{even} \xleftarrow{twice} \Phi_{even} \\
\equiv & \{ \ \ldots\ldots\ldots \ \} \\
 & twice \cdot \Phi_{even} \subseteq \Phi_{even} \cdot twice \\
\equiv & \{ \ \ldots\ldots\ldots \ \} \\
 & twice \cdot \Phi_{even} \subseteq \rho\ twice \cdot twice \\
\equiv & \{ \ \ldots\ldots\ldots \ \} \\
 & twice \cdot \Phi_{even} \subseteq twice \\
\Leftarrow & \{ \ \ldots\ldots\ldots \ \} \\
 & \Phi_{even} \subseteq id \\
\equiv & \{ \ \ldots\ldots\ldots \ \} \\
 & \textsc{True}
\end{array}
$$

□

Exercises

Exercise 65: Prove the **union simplicity** rule:

$$M \cup N \text{ is simple} \quad \equiv \quad M,\ N \text{ are simple and } M \cdot N^{\circ} \subseteq id \tag{160}$$

□