UCE: MFES-11/12

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CSI Module — Exercise list

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NB: Equation numbers of the form ([1]:n) are taken from [1].

Exercise 1. Check **carefully** which rules of the quantifier calculus need to be applied to prove that predicate

$$\langle \forall\, b, a \;:\; \langle \exists\, c \;:\; b = f\ c :\; r(c,a) \rangle :\; s(b,a) \rangle \tag{1}$$

is the same as

$$\langle \forall\, c, a \;:\; r(c,a) :\; s(f\ c, a) \rangle \tag{2}$$

where f is a function and r, s are binary predicates.
□

Exercise 2. Define relations $C \xleftarrow{R} A$, $A \xleftarrow{S} B$ such that $cRa = r(c,a)$ and $bSa = s(b,a)$. Then PF-transform (1) and (2), showing that the equivalence proved above is nothing but the rule

$$f \cdot R \subseteq S \Leftrightarrow R \subseteq f^\circ \cdot S \tag{3}$$

which is number ([1]:67) in the tutorial. □

Exercise 3. Given a function $B \xleftarrow{f} A$, show that *img* f is the coreflexive Φ_p of predicate $p\ x \triangleq \langle \exists\, a \;::\; x = f\ a \rangle$.
□

Exercise 4. Justify the following PF calculation of ([1]:67), where the equivalence is proved by cyclic implication ("ping-pong"):

$$
\begin{array}{ll}
& f \cdot R \subseteq S \\
\Rightarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f^\circ \cdot f \cdot R \subseteq f^\circ \cdot S \\
\Rightarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& R \subseteq f^\circ \cdot S \\
\Rightarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot R \subseteq f \cdot f^\circ \cdot S \\
\Rightarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot R \subseteq S
\end{array}
$$

□

Exercise 5. So, for f entire and simple ($\Leftrightarrow$ a function) rule ([1]:67) holds. Now, suppose that rule ([1]:67) holds for f replaced by an arbitrary relation X:

$$X \cdot R \subseteq S \Leftrightarrow R \subseteq X^\circ \cdot S \tag{4}$$

Check what you can infer about this rule for the particular instantiations:

- $R, S := id, X$ (left-cancellation)
- $S, R := id, X^\circ$ (right-cancellation)

Conclude that (4) holds **if and only if** X is a function.
□

Exercise 6. It can be shown that there is a relational operator known as relation division and denoted by $R \setminus S$, which satisfies the following equivalence, cf. ([1]:23):

$$R \cdot X \subseteq S \Leftrightarrow X \subseteq R \setminus S$$

Below we show that $R \setminus S$ hides a universal quantifier.

Draw a diagram rendering the types of X, R and S above explicit and complete the hints below:

$$
\begin{array}{ll}
& x\ (R \setminus S)\ y \\
\Leftrightarrow & \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \langle \forall\ c, d\ :\ c = \underline{x}\ d :\ c\ (R \setminus S)\ (\underline{y}\ d) \rangle \\
\Leftrightarrow & \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \underline{x} \subseteq (R \setminus S) \cdot \underline{y} \\
\Leftrightarrow & \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \underline{x} \cdot \underline{y}^\circ \subseteq R \setminus S \\
\Leftrightarrow & \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& R \cdot \underline{x} \subseteq S \cdot \underline{y} \\
\Leftrightarrow & \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \langle \forall\ b, a\ :\ b(R \cdot \underline{x})a :\ b(S \cdot \underline{y})a \rangle \\
\Leftrightarrow & \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \langle \forall\ b, a\ :\ b\ R\ (\underline{x}\ a) :\ b\ S\ (\underline{y}\ a) \rangle \\
\Leftrightarrow & \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \langle \forall\ b\ :\ b\ R\ x :\ b\ S\ y \rangle
\end{array}
$$

Thus

$$x\ (R \setminus S)\ y \Leftrightarrow \langle \forall\ b\ :\ b\ R\ x :\ b\ S\ y \rangle \tag{5}$$

References

1. J.N. Oliveira. *Extended Static Checking by Calculation using the Pointfree Transform* . In A. Bove et al., editor, *LerNet ALFA Summer School 2008*, volume 5520 of *LNCS*, pages 195–251. Springer-Verlag, 2009.