Structure and behaviour (background)

Luís S. Barbosa

HASLab - INESC TEC
Universidade do Minho
Braga, Portugal

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The architecture of **functional designs**

SA as studied at MFES (until now):

Interfaces:	$f :: \cdots \longrightarrow \cdots$
Components:	$f = \cdots$
Connectors:	$\cdot$, $\langle\ ,\ \rangle$, $\times$, $+$, ...
Configurations:	functions assembled by composition
Properties:	invariants (pre-, post-conditions)
Behavioural effects:	monads and Kleisli compostion
Underlying maths:	universal algebra and relational calculus

... can be extended to reactive systems?

Software Architecture is challenged by the continuous evolution towards very large, heterogeneous, highly dynamic computing systems, whose behaviour cannot be characterized in terms of a io-relation. In most cases, such a behaviour

- is potentially non-terminating,
- expresses a continued interaction with the system's environment and sub-systems which execute concurrently in distributed, often loosely coupled configurations.

Behaviour & Interaction

[R. Milner, 1997]
Thus software, from being a prescription for how to do something — in Turing's terms a "list of instructions" — becomes much more akin to a description of behaviour, not only programmed on a computer, but occurring by hap or design inside or outside it.

[B. Jacobs, 2005]
The subject of Computer Science is not information processing or symbol manipulation, but generated behaviour.

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Reactive systems

Reactive system

$\boxed{\text{system that computes by reacting to stimuli from its environment along its overall computation}}$

- in contrast to sequential systems whose meaning is defined by the results of finite computations, the behaviour of reactive systems is mainly determined by interaction and mobility of non-terminating processes, evolving concurrently.
- observation $\Leftrightarrow$ interaction
- behaviour $\Leftrightarrow$ a structured record of interactions

The architecture of **reactive designs**

Interfaces:	$\cdots$
Components:	$\cdots$
Connectors:	$\cdots$
Configurations:	$\cdots$
Properties:	$\cdots$
Behavioural effects:	$\cdots$
Underlying maths:	$\cdots$

Automata

Definition

$\mathcal{A} = \langle A, U, u_o, F, T \rangle$
where

- A is an alphabet
- $U = \{u_0, u_1, u_2, ...\}$ is a set of states
- $u_0 \in U$ is the initial state
- $F \subseteq U$ is the set of final states
- $T \subseteq U \times A \times U$ is the transition relation usually given as a A-indexed family of relations over U:

$$u' \xleftarrow{a} u \Leftrightarrow \langle u', a, u \rangle \in T$$

- deterministic
- finite
- image finite

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Automata

$$\boxed{\text{automaton behaviour} \quad \Leftrightarrow \quad \text{accepted language}}$$

Recall that finite automata recognize regular languages, i.e. generated by

- $L_1 + L_2 \;\triangleq\; L_1 \cup L_2$ (union)
- $L_1 \cdot L_2 \;\triangleq\; \{st \mid s \in L_1, t \in L_2\}$ (concatenation)
- $L^* \;\triangleq\; \{\epsilon\} \cup L \cup (L \cdot L) \cup (L \cdot L \cdot L) \cup \ldots$ (iteration)

Automata

There is a syntax to specify such languages:

$$E \quad ::= \quad \epsilon \mid a \mid E + E \mid E\,E \mid E^*$$

where $a \in \Sigma$.

- which regular expression specifies $\{a, bc\}$?
- and $\{ca, cb\}$?

and an algebra of regular expressions:

$$\begin{aligned}(E_1 + E_2) + E_3 &= E_1 + (E_2 + E_3)\\ (E_1 + E_2)\,E_3 &= E_1\,E_3 + E_2\,E_3\\ E_1\,(E_2\,E_1)^* &= (E_1\,E_2)^*\,E_1\end{aligned}$$

Automata

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After thoughts

... need more general models and theories:

- Several interaction points ($\neq$ functions)
- Non termination (no final states as in automata)
- Need to distinguish normal from anomolous termination (eg deadlock)
- Non determinisim should be taken seriously: the notion of equivalence based on accepted language is blind wrt non determinism

Labelled Transition Structure

Relational characterization

A LTS over a set A of names is a pair

$$\langle U, \ {}_{\alpha}\!\longleftarrow : A \times U \longleftarrow U \rangle$$

where

- $U = \{u_0, u_1, u_2, \ldots\}$ is a set of states
- ${}_{\alpha}\!\longleftarrow : A \times U \longleftarrow U$ is an arrow in Rel, capturing the transition relation,

  often given as an A-indexed family of binary relations

$$u' \ {}_{\alpha}\!\overset{a}{\longleftarrow} \ u \ \Leftrightarrow \ \langle a, u' \rangle \ {}_{\alpha}\!\longleftarrow \ u$$

or simply,

$$u' \ \overset{a}{\longleftarrow} \ u \ \Leftrightarrow \ \langle a, u' \rangle \ {}_{\alpha}\!\longleftarrow \ u$$

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Labelled Transition System

Definition

Given a transition structure $\langle U, \ _{\alpha}\!\longleftarrow : A \times U \longleftarrow U \rangle$, each $u \in U$ determines a labelled transition system, i.e.,

$$\langle U, u, \ _{\alpha}\!\longleftarrow : A \times U \longleftarrow U \rangle$$

fixing u as the initial state

Labelled Transition System

Pre- and Post-states

$$\mathrm{Post}_\alpha(u,a) = \{u' \in U \mid u' \;{}_\alpha\!\overset{a}{\Longleftarrow}\; u\} \qquad \mathrm{Post}_\alpha(u) = \bigcup_{a\in A} \mathrm{Post}_\alpha(u,a)$$

$$\mathrm{Pre}_\alpha(u,a) = \{u' \in U \mid u \;{}_\alpha\!\overset{a}{\Longleftarrow}\; u'\} \qquad \mathrm{Pre}_\alpha(u) = \bigcup_{a\in A} \mathrm{Pre}_\alpha(u,a)$$

Terminal state and successor

- u is a terminal state if $\mathrm{Post}_\alpha(u) = \emptyset$
- a a-successor is $\cdots$
- a direct successor is $\cdots$

Reachability

Definition

The reachability relation, ${}_{\alpha}\Longleftarrow : A \times U \longleftarrow U$, is defined inductively

- $u \;{}_{\alpha}\overset{\epsilon}{\Longleftarrow}\; u$ for each $u \in U$, where $\epsilon \in A^*$ denotes the empty word;
- if $u'' \;{}_{\alpha}\overset{\sigma}{\Longleftarrow}\; u$ and $u' \;{}_{\alpha}\overset{a}{\longleftarrow}\; u''$ then $u'' \;{}_{\alpha}\overset{\sigma a}{\Longleftarrow}\; u$, for $a \in A, \sigma \in A^*$

Reachable state

$t \in U$ is reachable from $u \in U$ iff there is a word $\sigma \in A^*$ st $t \;{}_{\alpha}\overset{\sigma}{\Longleftarrow}\; u$

Transposition

The power-transpose

Binary relations and powerset valued functions are equivalent: each other determines the other uniquely.
The existence and uniqueness of such a transformation leads to the identification of a transpose operator Λ characterized by the following universal property:

$$f = \Lambda R \;\Leftrightarrow\; (yRx \Leftrightarrow y \in f\,x)$$

for relation $R : Y \longleftarrow X$ and function $f : \mathcal{P}Y \longleftarrow X$ or, in a completely pointfree formulation

$$\boxed{f = \Lambda R \;\Leftrightarrow\; R = \in \cdot f}$$

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Transposition

$$f = \Lambda R \;\Leftrightarrow\; R = \in \cdot f$$

Properties

Cancellation	$\in \cdot \Lambda R \;=\; R$
Reflexivity	$\Lambda \in \;=\; \mathrm{id}$
Fusion - a	$\Lambda(f \cdot R) \;=\; \mathcal{P}f \cdot \Lambda R$
Fusion - b	$\Lambda(R \cdot f) \;=\; \Lambda R \cdot f$

Labelled Transition System

Transposing $_\alpha\!\longleftarrow$

through

$$\alpha = \Lambda \; {}_\alpha\!\longleftarrow \quad \Leftrightarrow \quad {}_\alpha\!\longleftarrow = \in \cdot \alpha$$

gives rise to function

$$\alpha : \mathcal{P}(A \times U) \longleftarrow U$$

Labelled Transition System

Transposing ${}_{\alpha}\!\longleftarrow$

through

$$\alpha = \Lambda \ {}_{\alpha}\!\longleftarrow \quad \Leftrightarrow \quad {}_{\alpha}\!\longleftarrow = \in \cdot \alpha$$

gives rise to function

$$\alpha : \mathcal{P}(A \times U) \longleftarrow U$$

Labelled Transition System

Transposition also applies to morphisms

A morphism $h : \beta \longleftarrow \alpha$ is a function $h : V \longleftarrow U$ st the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathcal{P}(A \times U) \arrow[d, "\mathcal{P}(\mathrm{id} \times h)"] \\ V \arrow[r, "\beta"] & \mathcal{P}(A \times V) \end{tikzcd}$$

i.e.,

$$\mathcal{P}(\mathrm{id} \times h) \cdot \alpha \;=\; \beta \cdot h$$

or, going pointwise,

$$\{\langle a, h\,x\rangle \mid \langle a, x\rangle \in \alpha\,u\} \;=\; \beta\,(h\;u)$$

Labelled Transition System

but $\mathcal{P}(\mathrm{id} \times h) \cdot \alpha \quad = \quad \beta \cdot h$

has the following relational counterpart:

$$\boxed{(\mathrm{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \quad = \quad {}_{\beta}\!\longleftarrow \cdot h}$$

because

Labelled Transition System

$$
\begin{array}{ll}
 & (\mathrm{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \;=\; {}_{\beta}\!\longleftarrow \cdot h \\
\Leftrightarrow & \quad \{ \text{ transpose is a isomorphism } \} \\
 & \Lambda((\mathrm{id} \times h) \cdot {}_{\alpha}\!\longleftarrow) \;=\; \Lambda({}_{\beta}\!\longleftarrow \cdot h) \\
\Leftrightarrow & \quad \{ \Lambda(f \cdot R) = \mathcal{P}f \cdot \Lambda R \ \text{ e } \ \Lambda(R \cdot f) = \Lambda R \cdot f \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda({}_{\alpha}\!\longleftarrow) \;=\; \Lambda({}_{\beta}\!\longleftarrow) \cdot h \\
\Leftrightarrow & \quad \{ \text{ definition } {}_{\alpha}\!\longleftarrow \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda(\in \cdot \alpha) \;=\; \Lambda(\in \cdot \beta) \cdot h \\
\Leftrightarrow & \quad \{ \Lambda(R \cdot f) = \Lambda R \cdot f \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda(\in) \cdot \alpha \;=\; \Lambda(\in) \cdot \beta \cdot h \\
\Leftrightarrow & \quad \{ \Lambda(\in) = \mathrm{id} \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \alpha \;=\; \beta \cdot h
\end{array}
$$

Labelled Transition System

Equality

$$(\mathrm{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \;=\; {}_{\beta}\!\longleftarrow \cdot h$$

can be re-written in terms of an A-indexed family of binary relations:

$$h \cdot {}_{\alpha}\!\xleftarrow{a} \;=\; {}_{\beta}\!\xleftarrow{a} \cdot h$$

which can be decomposed in

$$h \cdot {}_{\alpha}\!\xleftarrow{a} \;\subseteq\; {}_{\beta}\!\xleftarrow{a} \cdot h \qquad (1)$$

$${}_{\beta}\!\xleftarrow{a} \cdot h \;\subseteq\; h \cdot {}_{\alpha}\!\xleftarrow{a} \qquad (2)$$

Labelled Transition System

Equality

$$(\mathsf{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \;=\; {}_{\beta}\!\longleftarrow \cdot h$$

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$${}_{\beta}\!\xleftarrow{a} \cdot h \;\subseteq\; h \cdot {}_{\alpha}\!\xleftarrow{a} \tag{2}$$

Going pointwise ...

Transition preservation

$$
\begin{array}{cl}
 & h \cdot {}_{\alpha}\!\xleftarrow{a} \;\subseteq\; {}_{\beta}\!\xleftarrow{a} \cdot h \\
\Leftrightarrow & \quad \{ \; \textit{shunting} \; \} \\
 & {}_{\alpha}\!\xleftarrow{a} \;\subseteq\; h^{\circ} \cdot {}_{\beta}\!\xleftarrow{a} \cdot h \\
\Leftrightarrow & \quad \{ \; \text{introducing variables} \; \} \\
 & \langle \forall\, u, u' \;:\; u, u' \in U :\; u' \;{}_{\alpha}\!\xleftarrow{a}\; u \;\Rightarrow\; u' \,(h^{\circ} \cdot {}_{\beta}\!\xleftarrow{a} \cdot h)\, u \rangle \\
\Leftrightarrow & \quad \{ \; \text{relating-functional-images rule} \; \} \\
 & \langle \forall\, u, u' \;:\; u, u' \in U :\; u' \;{}_{\alpha}\!\xleftarrow{a}\; u \;\Rightarrow\; h\, u' \;{}_{\beta}\!\xleftarrow{a}\; h\, u \rangle
\end{array}
$$

Going pointwise ...

Transition reflection

$$
\begin{array}{ll}
 & {}_{\beta}\overset{a}{\longleftarrow} \cdot h \;\subseteq\; h \cdot {}_{\alpha}\overset{a}{\longleftarrow} \\
\Leftrightarrow & \quad \{ \text{ introducing variables } \} \\
 & \langle \forall\, u, v' \;:\; u \in U, v' \in V :\; v' \,({}_{\beta}\overset{a}{\longleftarrow} \cdot h)\, u \;\Rightarrow\; v' \,(h \cdot {}_{\alpha}\overset{a}{\longleftarrow})\, u \rangle \\
\Leftrightarrow & \quad \{ \text{ relating-functional-images rule and relational composition } \} \\
 & \langle \forall\, u, v' \;:\; u \in U, v' \in V :\; v' \;{}_{\beta}\overset{a}{\longleftarrow}\; h\, u \;\Rightarrow \\
 & \qquad\qquad \langle \exists\, u' \;:\; u' \in U :\; u' \;{}_{\alpha}\overset{a}{\longleftarrow}\; u \;\wedge\; v' = h\, u' \rangle\rangle
\end{array}
$$

Simulation

Intuition

A state v simulates another state u if every transition from v is corresponded by a transition from u and this capacity is kept along the whole life of the system to which state space v belongs to.

Simulation

Definition

Given ${}_{\alpha}\!\longleftarrow\, : U \times A \longleftarrow U$ and ${}_{\beta}\!\longleftarrow\, : V \times A \longleftarrow V$ both over A, a **simulation** of ${}_{\alpha}\!\longleftarrow$ in ${}_{\beta}\!\longleftarrow$ is a relation $S : V \longleftarrow U$ such that

$$\forall_{a \in A} \forall_{u \in U, v \in V} \,.\, vSu \;\Rightarrow$$
$$(\forall_{u' \in U} \,.\, u' \;{}_{\alpha}\!\xleftarrow{\;a\;}\; u \Rightarrow (\exists_{v' \in V} \,.\, v' \;{}_{\beta}\!\xleftarrow{\;a\;}\; v \wedge v' S u'))$$

$$\begin{array}{ccc} v & \Leftarrow & v \;\; S \;\; u \\ \downarrow a & & \downarrow a \\ v' \;\; S \;\; u' & & u' \end{array}$$

Simulation

Definition

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$$\begin{tikzcd}
v \arrow[d, "a"] & & & \Leftarrow & & v & S & u \arrow[d, "a"] \\
v' & S & u' & & & & & u'
\end{tikzcd}$$

Example

$$
\begin{tikzcd}
 & q_1 \arrow[r, "d"] & q_2 \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_4 \arrow[r, "e"'] & q_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & p_2 \\
p_0 \arrow[r, "a"] & p_1 \arrow[ur, "d"] \arrow[dr, "e"'] & \\
 & & p_3
\end{tikzcd}
$$

$q_0 \lesssim p_0$ cf. $\{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$

Simulation

Lemma

A relation $S : V \longleftarrow U$ is a simulation of $_{\alpha}\!\longleftarrow$ in $_{\beta}\!\longleftarrow$ iff, for all $a \in A$

$$S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S$$

Properties

because

$$
\begin{array}{ll}
& \forall_{a \in A, u \in U, v \in V} \,.\, vSu \;\Rightarrow \\
& \quad (\forall_{u' \in U} \,.\, u' \;{}_{\alpha}\!\xleftarrow{a}\; u \Rightarrow (\exists_{v' \in V} \,.\, v' \;{}_{\beta}\!\xleftarrow{a}\; v \wedge v' S u')) \\
\Leftrightarrow & \quad \{ \text{ composition } \} \\
& \forall_{a \in A, u \in U, v \in V} \,.\, vSu \;\Rightarrow\; (\forall_{u' \in U} \,.\, u \xrightarrow{a}_{\alpha} u' \Rightarrow v\,(\xrightarrow{a}_{\beta} \cdot S)\, u' \\
\Leftrightarrow & \quad \{ \text{ left relational division } \} \\
& \forall_{a \in A, u \in U, v \in V} \,.\, vSu \;\Rightarrow\; v\,((\xrightarrow{a}_{\beta} \cdot S) / \xrightarrow{a}_{\alpha})\, u \\
\Leftrightarrow & \quad \{ \text{ going pointfree } \} \\
& S \subseteq (\xrightarrow{a}_{\beta} \cdot S) / \xrightarrow{a}_{\alpha} \\
\Leftrightarrow & \quad \{ \text{ Galois connection: } (\cdot R) \dashv (/R) \; \} \\
& S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S
\end{array}
$$

Properties

Lemma

1. The identity relation id and the empty relation is a simulation
2. The composition $S \cdot R$ of two simulations is a simulation
3. The union $S \cup R$ of two simulations is a simulation

Properties

because

1.

$$\perp \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot \perp \quad \wedge \quad \mathsf{id} \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\alpha} \cdot \mathsf{id}$$

$\Leftrightarrow$ { $\perp$ and id are absorbing and identity for composition }

true

2.

$$(S \cdot R) \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot (S \cdot R)$$

$\Leftarrow$ { $S \cdot \xrightarrow{a}_{\gamma} \subseteq \xrightarrow{a}_{\beta} \cdot S$, $\cdot$-assoc, monotony }

$$(S \cdot R) \cdot \xrightarrow{a}_{\alpha} \subseteq S \cdot \xrightarrow{a}_{\gamma} \cdot R$$

$\Leftarrow$ { $R \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\gamma} \cdot R$, $\cdot$-assoc, monotony }

$$(S \cdot R) \cdot \xrightarrow{a}_{\alpha} \subseteq (S \cdot R) \cdot \xrightarrow{a}_{\alpha}$$

Properties

because

1.

$$\bot \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot \bot \;\wedge\; \mathrm{id} \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\alpha} \cdot \mathrm{id}$$

$\Leftrightarrow$ { $\bot$ and id are absorbing and identity for composition }

true

2.

$$(S \cdot R) \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot (S \cdot R)$$

$\Leftarrow$ { $S \cdot \xrightarrow{a}_{\gamma} \subseteq \xrightarrow{a}_{\beta} \cdot S$, $\cdot$-assoc, monotony }

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$$(S \cdot R) \cdot \xrightarrow{a}_{\alpha} \subseteq (S \cdot R) \cdot \xrightarrow{a}_{\alpha}$$

Properties

3.

$$
\begin{array}{ll}
 & (S \cup R) \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot (S \cup R) \\
\Leftrightarrow & \qquad \{ \ (R\cdot) \text{ and } (\cdot R) \text{ preserve } \cup \text{ as lower adjoints } \} \\
 & (S \cdot \xrightarrow{a}_{\alpha} \cup R \cdot \xrightarrow{a}_{\alpha}) \subseteq (\xrightarrow{a}_{\beta} \cdot S \cup \xrightarrow{a}_{\beta} \cdot R) \\
\Leftarrow & \qquad \{ \ \cup \text{ definition } \} \\
 & S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S \ \wedge \ R \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot R \\
\Leftrightarrow & \qquad \{ \ \text{hipotheses } \} \\
 & \text{true}
\end{array}
$$

Similarity

Definition

$$p \precsim q \;\Leftrightarrow\; \langle \exists\, R \;::\; R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

Lemma

The similarity relation is a preorder
(ie, reflexive and transitive)

because

By definition $\precsim$ is the greatest simulation. Then (why?), $\precsim \cdot \precsim \subseteq \precsim$ and id $\subseteq \precsim$.

Bisimulation

Definition

A relation $S : V \longleftarrow U$ over the state spaces of ${}_{\alpha}\!\longleftarrow : U \times A \longleftarrow U$ and ${}_{\beta}\!\longleftarrow : V \times A \longleftarrow V$ is a bisimulation iff both S and S° are simulations

i.e.

$$S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S \quad \wedge \quad {}_{\beta}\!\xleftarrow{a} \cdot S \subseteq S \cdot {}_{\alpha}\!\xleftarrow{a}$$

for all $a \in A$.

Bisimulation

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i.e.

$$\boxed{S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S \quad \wedge \quad {}_{\beta}\!\xleftarrow{a} \cdot S \subseteq S \cdot {}_{\alpha}\!\xleftarrow{a}}$$

for all $a \in A$.

Bisimulation

because
The first conjunct defines S as a simulation.
The second one is derived as follows:

$$
\begin{array}{ll}
& S^\circ \quad \text{is a simulation} \\
\Leftrightarrow & \qquad \{ \text{ definition of simulation } \} \\
& S^\circ \cdot \xrightarrow{a}_\beta \subseteq \xrightarrow{a}_\alpha \cdot S^\circ \\
\Leftrightarrow & \qquad \{ \ (\xrightarrow{a}_\gamma)^\circ = {}_\gamma\xleftarrow{a} \ \} \\
& S^\circ \cdot ({}_\beta\xleftarrow{a})^\circ \subseteq ({}_\alpha\xleftarrow{a})^\circ \cdot S^\circ \\
\Leftrightarrow & \qquad \{ \ (R \cdot S)^\circ = S^\circ \cdot R^\circ \ \} \\
& ({}_\beta\xleftarrow{a} \cdot S)^\circ \subseteq (S \cdot {}_\alpha\xleftarrow{a})^\circ \\
\Leftrightarrow & \qquad \{ \text{ monotonicity: } R \subseteq S \Leftrightarrow R^\circ \subseteq S^\circ \ \} \\
& {}_\beta\xleftarrow{a} \cdot S \subseteq S \cdot {}_\alpha\xleftarrow{a}
\end{array}
$$

Bisimulation

going pointwise

$$
\begin{array}{ll}
 & {}_{\beta}\xleftarrow{a} \cdot S \subseteq S \cdot {}_{\alpha}\xleftarrow{a} \\
\Leftrightarrow & \quad \{ \text{ Galois: } (R\cdot) \dashv (R\backslash) \ \} \\
 & S \subseteq {}_{\beta}\xleftarrow{a} \backslash (S \cdot {}_{\alpha}\xleftarrow{a}) \\
\Leftrightarrow & \quad \{ \text{ introducing variables } \} \\
 & \forall_{v \in V, u \in U} \,.\, vSu \Rightarrow v\,({}_{\beta}\xleftarrow{a} \backslash (S \cdot {}_{\alpha}\xleftarrow{a}))\,u \\
\Leftrightarrow & \quad \{ \text{ definition of left division } \backslash \ \} \\
 & \forall_{v \in V, u \in U} \,.\, vSu \Rightarrow (\forall_{v' \in V} \,.\, v' \, {}_{\alpha}\xleftarrow{a} \, v \Rightarrow v'\,({}_{\beta}\xleftarrow{a} \cdot S)\,u') \\
\Leftrightarrow & \quad \{ \text{ definition of } \cdot \ \} \\
 & \forall_{v \in V, u \in U} \,.\, vSu \Rightarrow (\forall_{v' \in V} \,.\, v' \, {}_{\beta}\xleftarrow{a} \, v \Rightarrow (\exists_{u' \in U} \,.\, u' \, {}_{\alpha}\xleftarrow{a} \, u \wedge v'Su'))
\end{array}
$$

Bisimulation

The Game characterization

Two players R and I discuss whether the transition structures are mutually corresponding

- R starts by chosing a transition
- I replies trying to match it
- if I succeeds, R plays again
- R wins if I fails to find a corresponding match
- I wins if it replies to all moves from R and the game is in a configuration where all states have been visited or R can't move further. In this case is said that I has a wining strategy

Examples

$$\begin{tikzcd} & q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\ q_2 \arrow[rr, "c"] & & q_3 \arrow[loop right, "c"] \end{tikzcd} \qquad \begin{tikzcd} m \arrow[d, "a"'] \\ n \arrow[loop right, "c"] \end{tikzcd}$$

$$q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_3 \xrightarrow{a} \cdots \qquad \begin{tikzcd} h \arrow[loop right, "a"] \end{tikzcd}$$

Examples

$$\begin{tikzcd}
& q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
q_2 \arrow[rr, "c"] & & q_3 \arrow[loop right, "c"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
m \arrow[d, "a"'] \\
n \arrow[loop right, "c"]
\end{tikzcd}$$

$$\begin{tikzcd}
q_1 \arrow[r, "a"] & q_2 \arrow[r, "a"] & q_3 \arrow[r, "a"] & \cdots
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
h \arrow[loop right, "a"]
\end{tikzcd}$$

Properties

Lemma

The graph of a transition structure morphism $h : \beta \longleftarrow \alpha$, i.e., h itself regarded as a binary relation, is a bisimulation.

Properties

because (the other inclusion being trivial):

$$
\begin{array}{ll}
 & h \cdot {}_\alpha\!\xleftarrow{a} \;\subseteq\; {}_\beta\!\xleftarrow{a} \cdot h \\
\Leftrightarrow & \quad \{ \text{ shunting } \} \\
 & {}_\alpha\!\xleftarrow{a} \;\subseteq\; h^\circ \cdot {}_\beta\!\xleftarrow{a} \cdot h \\
\Leftrightarrow & \quad \{ \text{ monotonicity } \} \\
 & ({}_\alpha\!\xleftarrow{a})^\circ \;\subseteq\; (h^\circ \cdot {}_\beta\!\xleftarrow{a} \cdot h)^\circ \\
\Leftrightarrow & \quad \{ \text{ converse } \} \\
 & \xrightarrow{a}_\alpha \;\subseteq\; h^\circ \cdot \xrightarrow{a}_\beta \cdot h \\
\Leftrightarrow & \quad \{ \text{ shunting } \} \\
 & h \cdot \xrightarrow{a}_\alpha \;\subseteq\; \xrightarrow{a}_\beta \cdot h
\end{array}
$$

Properties

Lemma

The converse of a bisimulation $S : V \longleftarrow U$ is still a bissimulation.

Properties

because

S° is bisimulation

$\Leftrightarrow$ { definition of bisimulation }

$S^\circ \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\beta \cdot S^\circ \;\wedge\; {}_\beta\xleftarrow{a} \cdot S^\circ \subseteq S^\circ \cdot {}_\alpha\xleftarrow{a}$

$\Leftrightarrow$ { $(\xrightarrow{a}_\gamma)^\circ = {}_\gamma\xleftarrow{a}$ }

$S^\circ \cdot ({}_\alpha\xleftarrow{a})^\circ \subseteq ({}_\beta\xleftarrow{a})^\circ \cdot S^\circ \;\wedge\; (\xrightarrow{a}_\beta)^\circ \cdot S^\circ \subseteq S^\circ \cdot (\xrightarrow{a}_\alpha)^\circ$

$\Leftrightarrow$ { converse of composition }

$({}_\alpha\xleftarrow{a} \cdot S)^\circ \subseteq (S \cdot {}_\beta\xleftarrow{a})^\circ \;\wedge\; (S \cdot \xrightarrow{a}_\beta)^\circ \subseteq (\xrightarrow{a}_\alpha \cdot S)^\circ$

$\Leftrightarrow$ { monotonicity }

${}_\alpha\xleftarrow{a} \cdot S \subseteq S \cdot {}_\beta\xleftarrow{a} \;\wedge\; S \cdot \xrightarrow{a}_\beta \subseteq \xrightarrow{a}_\alpha \cdot S$

$\Leftrightarrow$ { hipothesis }

true

Bisimilarity

Definition

$$p \sim q \;\Leftrightarrow\; \langle \exists\, R \;::\; R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

Lemma

1. The identity relation id is a bisimulation
2. The empty relation $\perp$ is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \mid i \in I\}$ is a bisimulation

Bisimilarity

Lemma

The bisimilarity relation is an equivalence relation
(ie, reflexive, symmetric and transitive)

Lemma

The class of all bisimulations between two LTS has the structure of a complete lattice, ordered by set inclusion, whose top is the bisimilarity relation $\sim$.

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Bisimilarity

Warning

The bisimilarity relation $\sim$ is not the symmetric closure of $\lesssim$

Example

$$q_0 \lesssim p_0, \; p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$

$$\begin{tikzcd}
 & q_1 & \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_2 \arrow[r, "b"] & q_3
\end{tikzcd}
\qquad
\begin{tikzcd}
p_0 \arrow[r, "a"] & p_1 \arrow[r, "b"] & p_3
\end{tikzcd}$$

After thoughts

Similarity as the greatest simulation

$$\lesssim \triangleq \bigcup\{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

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cf relational translation of definitions
$\lesssim$ and $\sim$ as greatest fix points (Tarski's theorem)

After thoughts

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