

Structure and behaviour (background)

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The architecture of **functional designs**

SA as studied at **MFES (until now)**:

Interfaces: $f :: \dots \longrightarrow \dots$

Components: $f = \dots$

Connectors: $\cdot, \langle, \rangle, \times, +, \dots$

Configurations: functions assembled by composition

Properties: invariants (pre-, post-conditions)

Behavioural effects: monads and Kleisli composition

Underlying maths: universal algebra and relational calculus

... can be extended to **reactive systems**?

Software Architecture is challenged by the continuous evolution towards very large, heterogeneous, highly dynamic computing systems, whose behaviour cannot be characterized in terms of a io-relation. In most cases, such a behaviour

- is potentially **non-terminating**,
- expresses a **continued interaction** with the system's environment and sub-systems which execute **concurrently** in distributed, often loosely coupled configurations.

Behaviour & Interaction

[R. Milner, 1997]

Thus software, from being a prescription for how to do something — in Turing's terms a "list of instructions" — becomes much more akin to a description of behaviour, not only programmed on a computer, but occurring by hap or design inside or outside it.

[B. Jacobs, 2005]

The subject of Computer Science is not information processing or symbol manipulation, but generated behaviour.

Behaviour & Interaction

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Reactive systems

Reactive system

system that computes by reacting to stimuli from its environment along its overall computation

- in contrast to sequential systems whose meaning is defined by the results of finite computations, the behaviour of reactive systems is mainly determined by **interaction** and **mobility** of **non-terminating** processes, evolving **concurrently**.
- **observation** \Leftrightarrow interaction
- **behaviour** \Leftrightarrow a structured record of interactions

The architecture of **reactive designs**

Interfaces:	...
Components:	...
Connectors:	...
Configurations:	...
Properties:	...
Behavioural effects:	...
Underlying maths:	...

Automata

Definition

$$\mathcal{A} = \langle A, U, u_0, F, T \rangle$$

where

- A is an alphabet
- $U = \{u_0, u_1, u_2, \dots\}$ is a set of states
- $u_0 \in U$ is the initial state
- $F \subseteq U$ is the set of final states
- $T \subseteq U \times A \times U$ is the transition relation usually given as a A -indexed family of relations over U :

$$u' \xrightarrow{a} u \Leftrightarrow \langle u', a, u \rangle \in T$$

- deterministic
- finite
- image finite

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Automata

automaton behaviour \Leftrightarrow accepted language

Recall that finite automata recognize **regular** languages, i.e. generated by

- $L_1 + L_2 \triangleq L_1 \cup L_2$ (union)
- $L_1 \cdot L_2 \triangleq \{st \mid s \in L_1, t \in L_2\}$ (concatenation)
- $L^* \triangleq \{\epsilon\} \cup L \cup (L \cdot L) \cup (L \cdot L \cdot L) \cup \dots$ (iteration)

Automata

There is a **syntax** to specify such languages:

$$E ::= \epsilon \mid a \mid E + E \mid E E \mid E^*$$

where $a \in \Sigma$.

- which regular expression specifies $\{a, bc\}$?
- and $\{ca, cb\}$?

and an **algebra of regular expressions**:

$$(E_1 + E_2) + E_3 = E_1 + (E_2 + E_3)$$

$$(E_1 + E_2) E_3 = E_1 E_3 + E_2 E_3$$

$$E_1 (E_2 E_1)^* = (E_1 E_2)^* E_1$$

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After thoughts

... need more general models and theories:

- **Several interaction points** (\neq functions)
- **Non termination** (no final states as in automata)
- Need to distinguish **normal from anomolous termination** (eg deadlock)
- **Non determinisim** should be taken seriously: the notion of **equivalence** based on accepted language is **blind** wrt non determinism

Labelled Transition Structure

Relational characterization

A LTS over a set A of **names** is a pair

$$\langle U, \alpha \longleftarrow : A \times U \longleftarrow U \rangle$$

where

- $U = \{u_0, u_1, u_2, \dots\}$ is a **set of states**
- $\alpha \longleftarrow : A \times U \longleftarrow U$ is an arrow in Rel, capturing the **transition relation**,
often given as an A -indexed family of binary relations

$$u' \xleftarrow{\alpha^a} u \Leftrightarrow \langle a, u' \rangle \alpha \longleftarrow u$$

or simply,

$$u' \xleftarrow{a} u \Leftrightarrow \langle a, u' \rangle \alpha \longleftarrow u$$

Labelled Transition Structure

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Labelled Transition System

Definition

Given a transition structure $\langle U, \alpha \longleftarrow : A \times U \longleftarrow U \rangle$, each $u \in U$ determines a **labelled transition system**, i.e.,

$$\langle U, u, \alpha \longleftarrow : A \times U \longleftarrow U \rangle$$

fixing u as the **initial state**

Labelled Transition System

Pre- and Post-states

$$\text{Post}_\alpha(u, a) = \{u' \in U \mid u' \xrightarrow{a}_\alpha u\}$$

$$\text{Post}_\alpha(u) = \bigcup_{a \in A} \text{Post}_\alpha(u, a)$$

$$\text{Pre}_\alpha(u, a) = \{u' \in U \mid u \xrightarrow{a}_\alpha u'\}$$

$$\text{Pre}_\alpha(u) = \bigcup_{a \in A} \text{Pre}_\alpha(u, a)$$

Terminal state and successor

- u is a **terminal** state if $\text{Post}_\alpha(u) = \emptyset$
- a a -successor is \dots
- a direct successor is \dots

Reachability

Definition

The reachability relation, $\alpha \longleftarrow : A \times U \longleftarrow U$, is defined inductively

- $u \alpha \xleftarrow{\epsilon} u$ for each $u \in U$, where $\epsilon \in A^*$ denotes the empty word;
- if $u'' \alpha \xleftarrow{\sigma} u$ and $u' \alpha \xleftarrow{a} u''$ then $u'' \alpha \xleftarrow{\sigma a} u$, for $a \in A, \sigma \in A^*$

Reachable state

$t \in U$ is **reachable** from $u \in U$ iff there is a word $\sigma \in A^*$ st $t \alpha \xleftarrow{\sigma} u$

Transposition

The power-transpose

Binary **relations** and **powerset valued functions** are equivalent: each other determines the other **uniquely**.

The existence and uniqueness of such a transformation leads to the identification of a **transpose operator** Λ characterized by the following **universal property**:

$$f = \Lambda R \Leftrightarrow (yRx \Leftrightarrow y \in f x)$$

for relation $R : Y \longleftarrow X$ and function $f : \mathcal{P}Y \longleftarrow X$ or, in a completely pointfree formulation

$$f = \Lambda R \Leftrightarrow R = \in \cdot f$$

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Transposition

$$f = \Lambda R \Leftrightarrow R = \in \cdot f$$

Properties

Cancellation

$$\in \cdot \Lambda R = R$$

Reflexivity

$$\Lambda \in = \text{id}$$

Fusion - a

$$\Lambda(f \cdot R) = \mathcal{P}f \cdot \Lambda R$$

Fusion - b

$$\Lambda(R \cdot f) = \Lambda R \cdot f$$

Labelled Transition System

Transposing $\alpha \longleftarrow$

through

$$\alpha = \bigwedge \alpha \longleftarrow \Leftrightarrow \alpha \longleftarrow = \epsilon \cdot \alpha$$

gives rise to function

$$\alpha : \mathcal{P}(A \times U) \longleftarrow U$$

Labelled Transition System

Transposing $\alpha \longleftarrow$

through

$$\alpha = \bigwedge \alpha \longleftarrow \Leftrightarrow \alpha \longleftarrow = \epsilon \cdot \alpha$$

gives rise to function

$$\alpha : \mathcal{P}(A \times U) \longleftarrow U$$

Labelled Transition System

Transposition also applies to morphisms

A **morphism** $h : \beta \leftarrow \alpha$ is a function $h : V \leftarrow U$ st the following diagram commutes

$$\begin{array}{ccc}
 U & \xrightarrow{\alpha} & \mathcal{P}(A \times U) \\
 h \downarrow & & \downarrow \mathcal{P}(\text{id} \times h) \\
 V & \xrightarrow{\beta} & \mathcal{P}(A \times V)
 \end{array}$$

i.e.,

$$\mathcal{P}(\text{id} \times h) \cdot \alpha = \beta \cdot h$$

or, going pointwise,

$$\{\langle a, hx \rangle \mid \langle a, x \rangle \in \alpha u\} = \beta(hu)$$

Labelled Transition System

but $\mathcal{P}(\text{id} \times h) \cdot \alpha = \beta \cdot h$

has the following **relational** counterpart:

$$(\text{id} \times h) \cdot \alpha \longleftarrow = \beta \longleftarrow \cdot h$$

because

Labelled Transition System

$$(\text{id} \times h) \cdot \alpha \longleftarrow = \beta \longleftarrow \cdot h$$

$$\Leftrightarrow \{ \text{transpose is a isomorphism} \}$$

$$\Lambda((\text{id} \times h) \cdot \alpha \longleftarrow) = \Lambda(\beta \longleftarrow \cdot h)$$

$$\Leftrightarrow \{ \Lambda(f \cdot R) = \mathcal{P}f \cdot \Lambda R \quad \text{e} \quad \Lambda(R \cdot f) = \Lambda R \cdot f \}$$

$$\mathcal{P}(\text{id} \times h) \cdot \Lambda(\alpha \longleftarrow) = \Lambda(\beta \longleftarrow) \cdot h$$

$$\Leftrightarrow \{ \text{definition } \alpha \longleftarrow \}$$

$$\mathcal{P}(\text{id} \times h) \cdot \Lambda(\in \cdot \alpha) = \Lambda(\in \cdot \beta) \cdot h$$

$$\Leftrightarrow \{ \Lambda(R \cdot f) = \Lambda R \cdot f \}$$

$$\mathcal{P}(\text{id} \times h) \cdot \Lambda(\in) \cdot \alpha = \Lambda(\in) \cdot \beta \cdot h$$

$$\Leftrightarrow \{ \Lambda(\in) = \text{id} \}$$

$$\mathcal{P}(\text{id} \times h) \cdot \alpha = \beta \cdot h$$

Labelled Transition System

Equality

$$(\text{id} \times h) \cdot \alpha \longleftarrow = \beta \longleftarrow \cdot h$$

can be re-written in terms of an A -indexed family of binary relations:

$$h \cdot \alpha \longleftarrow^a = \beta \longleftarrow^a \cdot h$$

which can be decomposed in

$$h \cdot \alpha \longleftarrow^a \subseteq \beta \longleftarrow^a \cdot h \tag{1}$$

$$\beta \longleftarrow^a \cdot h \subseteq h \cdot \alpha \longleftarrow^a \tag{2}$$

Labelled Transition System

Equality

$$(\text{id} \times h) \cdot \alpha \longleftarrow = \beta \longleftarrow \cdot h$$

can be re-written in terms of an **A-indexed family of binary relations**:

$$h \cdot \alpha \xleftarrow{a} = \beta \xleftarrow{a} \cdot h$$

which can be decomposed in

$$h \cdot \alpha \xleftarrow{a} \subseteq \beta \xleftarrow{a} \cdot h \tag{1}$$

$$\beta \xleftarrow{a} \cdot h \subseteq h \cdot \alpha \xleftarrow{a} \tag{2}$$

Going pointwise ...

Transition preservation

$$h \cdot \alpha \xrightarrow{a} \subseteq \beta \xrightarrow{a} \cdot h$$

$$\Leftrightarrow \{ \textit{shunting} \}$$

$$\alpha \xrightarrow{a} \subseteq h^\circ \cdot \beta \xrightarrow{a} \cdot h$$

$$\Leftrightarrow \{ \textit{introducing variables} \}$$

$$\langle \forall u, u' : u, u' \in U : u' \alpha \xrightarrow{a} u \Rightarrow u' (h^\circ \cdot \beta \xrightarrow{a} \cdot h) u \rangle$$

$$\Leftrightarrow \{ \textit{relating-functional-images rule} \}$$

$$\langle \forall u, u' : u, u' \in U : u' \alpha \xrightarrow{a} u \Rightarrow h u' \beta \xrightarrow{a} h u \rangle$$

Going pointwise ...

Transition reflection

$$\beta \xleftarrow{a} \cdot h \subseteq h \cdot \alpha \xleftarrow{a}$$

\Leftrightarrow { introducing variables }

$$\langle \forall u, v' : u \in U, v' \in V : v' (\beta \xleftarrow{a} \cdot h) u \Rightarrow v' (h \cdot \alpha \xleftarrow{a}) u \rangle$$

\Leftrightarrow { relating-functional-images rule and relational composition }

$$\langle \forall u, v' : u \in U, v' \in V : v' \beta \xleftarrow{a} h u \Rightarrow \\ \langle \exists u' : u' \in U : u' \alpha \xleftarrow{a} u \wedge v' = h u' \rangle \rangle$$

Simulation

Intuition

A state v **simulates** another state u if every transition from v is corresponded by a transition from u and this capacity is kept along the whole life of the system to which state space v belongs to.

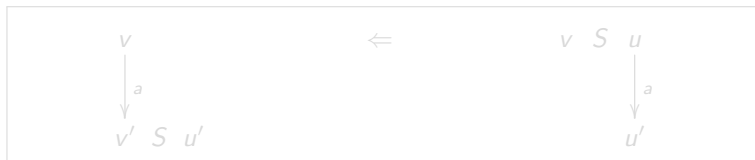
Simulation

Definition

Given $\alpha \leftarrow : U \times A \leftarrow U$ and $\beta \leftarrow : V \times A \leftarrow V$ both over A , a **simulation** of $\alpha \leftarrow$ in $\beta \leftarrow$ is a relation $S : V \leftarrow U$ such that

$$\forall a \in A \forall u \in U, v \in V. v S u \Rightarrow$$

$$(\forall u' \in U. u' \xrightarrow{a} u \Rightarrow (\exists v' \in V. v' \xrightarrow{a} v \wedge v' S u'))$$



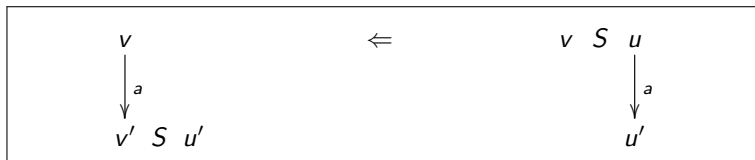
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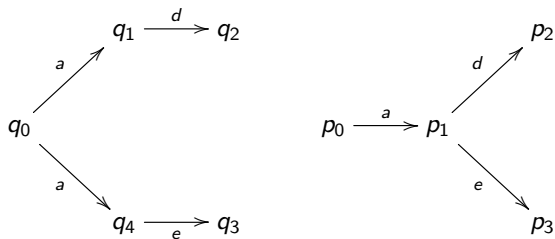
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Example



$$q_0 \lesssim p_0 \quad \text{cf.} \quad \{\langle q_0, p_0 \rangle, \langle q_1, p_1 \rangle, \langle q_4, p_1 \rangle, \langle q_2, p_2 \rangle, \langle q_3, p_3 \rangle\}$$

Simulation

Lemma

A relation $S : V \longleftarrow U$ is a **simulation** of $\alpha \longleftarrow$ in $\beta \longleftarrow$ iff, for all $a \in A$

$$S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S$$

Properties

because

$$\forall a \in A, u \in U, v \in V. vSu \Rightarrow$$

$$(\forall u' \in U. u' \xrightarrow{a} u \Rightarrow (\exists v' \in V. v' \xrightarrow{a} v \wedge v'Su'))$$

$$\Leftrightarrow \{ \text{composition} \}$$

$$\forall a \in A, u \in U, v \in V. vSu \Rightarrow (\forall u' \in U. u \xrightarrow{a} u' \Rightarrow v(\xrightarrow{a} \cdot S)u')$$

$$\Leftrightarrow \{ \text{left relational division} \}$$

$$\forall a \in A, u \in U, v \in V. vSu \Rightarrow v((\xrightarrow{a} \cdot S) / \xrightarrow{a})u$$

$$\Leftrightarrow \{ \text{going pointfree} \}$$

$$S \subseteq (\xrightarrow{a} \cdot S) / \xrightarrow{a}$$

$$\Leftrightarrow \{ \text{Galois connection: } (\cdot R) \dashv (/R) \}$$

$$S \cdot \xrightarrow{a} \subseteq \xrightarrow{a} \cdot S$$

Properties

Lemma

1. The identity relation id and the empty relation is a simulation
2. The composition $S \cdot R$ of two simulations is a simulation
3. The union $S \cup R$ of two simulations is a simulation

Properties

because

1.

$$\perp \cdot \xrightarrow{a} \alpha \subseteq \xrightarrow{a} \beta \cdot \perp \quad \wedge \quad \text{id} \cdot \xrightarrow{a} \alpha \subseteq \xrightarrow{a} \alpha \cdot \text{id}$$

$$\Leftrightarrow \quad \{ \perp \text{ and id are absorbing and identity for composition } \}$$

true

2.

$$(S \cdot R) \cdot \xrightarrow{a} \alpha \subseteq \xrightarrow{a} \beta \cdot (S \cdot R)$$

$$\Leftrightarrow \quad \{ S \cdot \xrightarrow{a} \gamma \subseteq \xrightarrow{a} \beta \cdot S, \text{ --assoc, monotony } \}$$

$$(S \cdot R) \cdot \xrightarrow{a} \alpha \subseteq S \cdot \xrightarrow{a} \gamma \cdot R$$

$$\Leftrightarrow \quad \{ R \cdot \xrightarrow{a} \alpha \subseteq \xrightarrow{a} \gamma \cdot R, \text{ --assoc, monotony } \}$$

$$(S \cdot R) \cdot \xrightarrow{a} \alpha \subseteq (S \cdot R) \cdot \xrightarrow{a} \alpha$$

Properties

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$$(S \cdot R) \cdot \xrightarrow{a}_{\alpha} \subseteq (S \cdot R) \cdot \xrightarrow{a}_{\alpha}$$

Properties

3.

$$\begin{aligned}
 & (S \cup R) \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot (S \cup R) \\
 \Leftrightarrow & \quad \{ (R \cdot) \text{ and } (\cdot R) \text{ preserve } \cup \text{ as lower adjoints} \} \\
 & (S \cdot \xrightarrow{a}_{\alpha} \cup R \cdot \xrightarrow{a}_{\alpha}) \subseteq (\xrightarrow{a}_{\beta} \cdot S \cup \xrightarrow{a}_{\beta} \cdot R) \\
 \Leftarrow & \quad \{ \cup \text{ definition} \} \\
 & S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S \quad \wedge \quad R \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot R \\
 \Leftrightarrow & \quad \{ \text{hypotheses} \} \\
 & \text{true}
 \end{aligned}$$

Similarity

Definition

$$p \lesssim q \Leftrightarrow \langle \exists R :: R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

Lemma

The similarity relation is a preorder
(ie, reflexive and transitive)

because

By definition \lesssim is the **greatest** simulation. Then (why?), $\lesssim \cdot \lesssim \subseteq \lesssim$ and $\text{id} \subseteq \lesssim$.

Bisimulation

Definition

A relation $S : V \longleftarrow U$ over the state spaces of $\alpha \longleftarrow : U \times A \longleftarrow U$ and $\beta \longleftarrow : V \times A \longleftarrow V$ is a **bisimulation** iff **both** S and S° are simulations
i.e.

$$S \cdot \xrightarrow{a} \alpha \subseteq \xrightarrow{a} \beta \cdot S \quad \wedge \quad \beta \xleftarrow{a} \cdot S \subseteq S \cdot \alpha \xleftarrow{a}$$

for all $a \in A$.

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for all $a \in A$.

Bisimulation

because

The first conjunct defines S as a simulation.

The second one is derived as follows:

S° is a simulation

$$\Leftrightarrow \{ \text{definition of simulation} \}$$

$$S^\circ \cdot \xrightarrow{a}_\beta \subseteq \xrightarrow{a}_\alpha \cdot S^\circ$$

$$\Leftrightarrow \{ (\xrightarrow{a}_\gamma)^\circ = \xleftarrow{a}_\gamma \}$$

$$S^\circ \cdot (\xleftarrow{a}_\beta)^\circ \subseteq (\xleftarrow{a}_\alpha)^\circ \cdot S^\circ$$

$$\Leftrightarrow \{ (R \cdot S)^\circ = S^\circ \cdot R^\circ \}$$

$$(\xleftarrow{a}_\beta \cdot S)^\circ \subseteq (S \cdot \xleftarrow{a}_\alpha)^\circ$$

$$\Leftrightarrow \{ \text{monotonicity: } R \subseteq S \Leftrightarrow R^\circ \subseteq S^\circ \}$$

$$\xleftarrow{a}_\beta \cdot S \subseteq S \cdot \xleftarrow{a}_\alpha$$

Bisimulation

going pointwise

$$\beta \xleftarrow{a} \cdot S \subseteq S \cdot \alpha \xleftarrow{a}$$

$$\Leftrightarrow \{ \text{Galois: } (R \cdot) \dashv (R \setminus) \}$$

$$S \subseteq \beta \xleftarrow{a} \setminus (S \cdot \alpha \xleftarrow{a})$$

$$\Leftrightarrow \{ \text{introducing variables} \}$$

$$\forall v \in V, u \in U. v S u \Rightarrow v (\beta \xleftarrow{a} \setminus (S \cdot \alpha \xleftarrow{a})) u$$

$$\Leftrightarrow \{ \text{definition of left division } \setminus \}$$

$$\forall v \in V, u \in U. v S u \Rightarrow (\forall v' \in V. v' \alpha \xleftarrow{a} v \Rightarrow v' (\beta \xleftarrow{a} \cdot S) u')$$

$$\Leftrightarrow \{ \text{definition of } \cdot \}$$

$$\forall v \in V, u \in U. v S u \Rightarrow (\forall v' \in V. v' \beta \xleftarrow{a} v \Rightarrow (\exists u' \in U. u' \alpha \xleftarrow{a} u \wedge v' S u'))$$

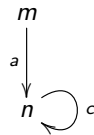
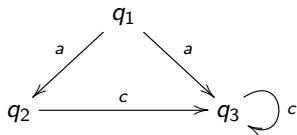
Bisimulation

The Game characterization

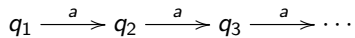
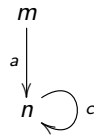
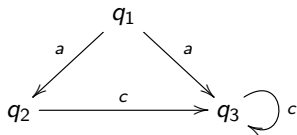
Two players R and I discuss whether the transition structures are mutually corresponding

- R starts by choosing a transition
- I replies trying to match it
- if I succeeds, R plays again
- R wins if I fails to find a corresponding match
- I wins if it replies to all moves from R and the game is in a configuration where all states have been visited or R can't move further. In this case is said that I has a **wining strategy**

Examples



Examples



Properties

Lemma

The graph of a transition structure morphism $h : \beta \longleftarrow \alpha$, i.e., h itself regarded as a binary relation, is a **bisimulation**.

Properties

because (the other inclusion being trivial):

$$\begin{aligned}
 & h \cdot \alpha \xleftarrow{a} \subseteq \beta \xleftarrow{a} \cdot h \\
 \Leftrightarrow & \quad \{ \text{shunting} \} \\
 & \alpha \xleftarrow{a} \subseteq h^\circ \cdot \beta \xleftarrow{a} \cdot h \\
 \Leftrightarrow & \quad \{ \text{monotonicity} \} \\
 & (\alpha \xleftarrow{a})^\circ \subseteq (h^\circ \cdot \beta \xleftarrow{a} \cdot h)^\circ \\
 \Leftrightarrow & \quad \{ \text{converse} \} \\
 & \xrightarrow{a} \alpha \subseteq h^\circ \cdot \xrightarrow{a} \beta \cdot h \\
 \Leftrightarrow & \quad \{ \text{shunting} \} \\
 & h \cdot \xrightarrow{a} \alpha \subseteq \xrightarrow{a} \beta \cdot h
 \end{aligned}$$

Properties

Lemma

The converse of a bisimulation $S : V \longleftarrow U$ is still a bisimulation.

Properties

because

S° is bisimulation

\Leftrightarrow { definition of bisimulation }

$$S^\circ \cdot \xrightarrow{a} \alpha \subseteq \xrightarrow{a} \beta \cdot S^\circ \quad \wedge \quad \beta \xleftarrow{a} \cdot S^\circ \subseteq S^\circ \cdot \alpha \xleftarrow{a}$$

\Leftrightarrow { $(\xrightarrow{a} \gamma)^\circ = \gamma \xleftarrow{a}$ }

$$S^\circ \cdot (\alpha \xleftarrow{a})^\circ \subseteq (\beta \xleftarrow{a})^\circ \cdot S^\circ \quad \wedge \quad (\xrightarrow{a} \beta)^\circ \cdot S^\circ \subseteq S^\circ \cdot (\xrightarrow{a} \alpha)^\circ$$

\Leftrightarrow { converse of composition }

$$(\alpha \xleftarrow{a} \cdot S)^\circ \subseteq (S \cdot \beta \xleftarrow{a})^\circ \quad \wedge \quad (S \cdot \xrightarrow{a} \beta)^\circ \subseteq (\xrightarrow{a} \alpha \cdot S)^\circ$$

\Leftrightarrow { monotonicity }

$$\alpha \xleftarrow{a} \cdot S \subseteq S \cdot \beta \xleftarrow{a} \quad \wedge \quad S \cdot \xrightarrow{a} \beta \subseteq \xrightarrow{a} \alpha \cdot S$$

\Leftrightarrow { hypothesis }

true

Bisimilarity

Definition

$$p \sim q \Leftrightarrow \langle \exists R :: R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

Lemma

1. The identity relation id is a bisimulation
2. The empty relation \perp is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \mid i \in I\}$ is a bisimulation

Bisimilarity

Lemma

The bisimilarity relation is an equivalence relation (ie, reflexive, symmetric and transitive)

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The class of all bisimulations between two LTS has the structure of a **complete lattice**, ordered by set inclusion, whose top is the **bisimilarity** relation \sim .

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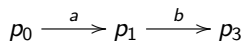
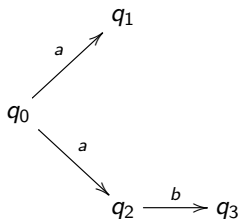
Bisimilarity

Warning

The bisimilarity relation \sim is not the symmetric closure of \lesssim

Example

$$q_0 \lesssim p_0, p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$



After thoughts

Similarity as the greatest simulation

$$\lesssim \triangleq \bigcup \{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

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cf relational translation of definitions
 \lesssim and \sim as greatest fix points (Tarski's theorem)

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