Introduction to process algebra

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Actions & processes

Action

is a latency for interaction

$$Act \quad ::= \quad a \mid \overline{a} \mid \tau$$

for $a \in L$, L denoting a set of names

Process

is a description of how the interaction capacities of a system evolve, *i.e.*, its behaviour
for example,

$$E \triangleq a.b.\mathbf{0} + a.E$$

- analogy: regular expressions vs finite automata

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Examples

Buffers

1-position buffer: $A(in, out) \triangleq in.\overline{out}.\mathbf{0}$

... non terminating: $B(in, out) \triangleq in.\overline{out}.B$

... with two output ports: $C(in, o_1, o_2) \triangleq in.(\overline{o_1}.C + \overline{o_2}.C)$

... non deterministic: $D(in, o_1, o_2) \triangleq in.\overline{o_1}.D + in.\overline{o_2}.D$

... with parameters: $B(in, out) \triangleq in(x).\overline{out}\langle x \rangle.B$

Parallel composition

n-position buffers

1-position buffer:

$$S \triangleq \text{new}\,\{m\}\;(B\langle in, m\rangle \mid B\langle m, out\rangle)$$

n-position buffer:

$$Bn \triangleq \text{new}\,\{m_i | i < n\}\;(B\langle in, m_1\rangle \mid B\langle m_1, m_2\rangle \mid \cdots \mid B\langle m_{n-1}, out\rangle)$$

Parallel composition

mutual exclusion

$$Sem \triangleq get.put.Sem$$

$$P_i \triangleq \overline{get}.c_i.\overline{put}.P_i$$

$$S \triangleq \text{new}\,\{get, put\}\ (Sem \mid (|_{i \in I}\ P_i))$$

A language for processes

Questions

- Which syntax to use to describe processes?
- What's the meaning of such descriptions?
- Why some of our favourite programming languages' constructions are not considered?
- ...

Syntax

The set $\mathbb{P}$ of processes is the set of all terms generated by the following BNF:

$$E \ ::= \ A(x_1, \dots, x_n) \ | \ a.E \ | \ \sum_{i \in I} E_i \ | \ E_0 \,|\, E_1 \ | \ \text{new}\, K \ E$$

for $a \in Act$ and $K \subseteq L$

Abbreviatures

$$E_0 + E_1 \stackrel{\text{abv}}{=} \sum_{i \in \{0,1\}} E_i$$

$$\mathbf{0} \stackrel{\text{abv}}{=} \sum_{i \in \emptyset} E_i$$

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Syntax

Process declaration

$$A(\tilde{x}) \triangleq E_A$$

with $\mathrm{fn}(E_A) \subseteq \tilde{x}$ (where $\mathrm{fn}(P)$ is the set of free variables of P).

- used as, *e.g.*, $\boxed{A(a, b, c) \triangleq a.b.\mathbf{0} + c.A\langle d, e, f\rangle}$

Process declaration: fixed point expression

$$\underline{\mathit{fix}}\ (X = E_X)$$

- syntactic substitution over $\mathbb{P}$, *cf.*,
  - $\{c/b\}\, a.b.\mathbf{0}$
  - (internal variables renaming)
    $\{x/y\}\ \mathrm{new}\ \{x\}\ y.x.\mathbf{0} = \mathrm{new}\ \{x'\}\ x.x'.\mathbf{0}$

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Sort

The sort of a process P is its interface, *i.e.*, its iteraction possibilities

- minimal sort: $\bigcap\{K \subseteq L \mid P : K\}$
- syntactic sort, *i.e.*, the set of free variables:

$$\begin{aligned}
\mathrm{fn}(a.P) &= \{a\} \cup \mathrm{fn}(P) \\
\mathrm{fn}(\tau.P) &= \mathrm{fn}(P) \\
\mathrm{fn}(\sum_{i\in I} P_i) &= \bigcup_{i\in I} \mathrm{fn}(P_i) \\
\mathrm{fn}(P \mid Q) &= \mathrm{fn}(P) \cup \mathrm{fn}(Q) \\
\mathrm{fn}(\mathrm{new}\, K\ P) &= \mathrm{fn}(P) - (K \cup \overline{K})
\end{aligned}$$

and, for each $P(\tilde{x}) \triangleq E$, $\mathrm{fn}(E) \subseteq \mathrm{fn}(P(\tilde{x})) = \tilde{x}$.

Sort

Warning

- new $\{a\}$ $(a.b.c.0)$ has no transitions, so its sort is $\emptyset$
- however: $\mathrm{fn}((\mathrm{new}\ \{a\}\ a.b.c.0)) = \{b, c\}$

Semantics

Two-level semantics

- arquitectural, expresses a notion of similar assembly configurations and is expressed through a structural congruence relation;
- comportamental given by transition rules which express how system's components interact

Semantics

Structural congruence

$\equiv$ over $\mathbb{P}$ is given by the closure of the following conditions:

- for all $A(\tilde{x}) \triangleq E_A$, $A(\tilde{y}) \equiv \{\tilde{x}/\tilde{y}\}\, E_A$,
  (*i.e.*, folding/unfolding preserve $\equiv$)
- α-conversion (*i.e.*, replacement of bounded variables).
- both $|$ and $+$ originate, with $\mathbf{0}$, abelian monoids
- forall $a \notin \mathrm{fn}(P)$ new $\{a\}$ $(P \mid Q) \equiv P \mid$ new $\{a\}$ Q
- new $\{a\}$ $\mathbf{0} \equiv \mathbf{0}$

Semantics

$$\frac{}{E \xleftarrow{a} a.E}\,(prefix)$$

$$\frac{E' \xleftarrow{a} \{\tilde{k}/\tilde{x}\}\, E_A}{E' \xleftarrow{a} A(\tilde{k})}\,(ident) \quad (\text{if } A(\tilde{x}) \triangleq E_A)$$

$$\frac{E' \xleftarrow{a} E}{E' \xleftarrow{a} E + F}\,(sum - l) \qquad \frac{F' \xleftarrow{a} F}{F' \xleftarrow{a} E + F}\,(sum - r)$$

Semantics

$$\frac{E' \xleftarrow{a} E}{E' \mid F \xleftarrow{a} E \mid F}\,(par - l) \qquad \frac{F' \xleftarrow{a} F}{E \mid F' \xleftarrow{a} E \mid F}\,(par - r)$$

$$\frac{E' \xleftarrow{a} E \quad F' \xleftarrow{\overline{a}} F}{E' \mid F' \xleftarrow{\tau} E \mid F}\,(react)$$

$$\frac{E' \xleftarrow{a} E}{\mathrm{new}\,\{k\}\; E' \xleftarrow{a} \mathrm{new}\,\{k\}\; E}\,(res) \quad (\text{if } a \notin \{k, \overline{k}\})$$

Compatibility

Lemma

Structural congruence preserves transitions:

if $E' \xleftarrow{a} E$ and $E \equiv F$ there exists a process F' such that $F' \xleftarrow{a} F$ and $E' \equiv F'$.

Semantics

These rules define a LTS

$$\{\xleftarrow{a} \subseteq \mathbb{P} \times \mathbb{P} \mid a \in Act\}$$

Relation $\xleftarrow{a}$ is defined inductively over process structure entailing a semantic description which is

Structural *i.e.*, each process shape (defined by the most external combinator) has a type of transitions

Modular *i.e.*, a process trasition is defined from transitions in its sup-processes

Complete *i.e.*, all possible transitions are infered from these rules

static vs dynamic combinators

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static vs dynamic combinators

Graphical representations

Synchronization diagram

- represent interfaces of processes
- static combinators are an algebra of synchronization diagrams

Transition graph

- derivative, n-derivative, transition tree
- folds into a transition graph

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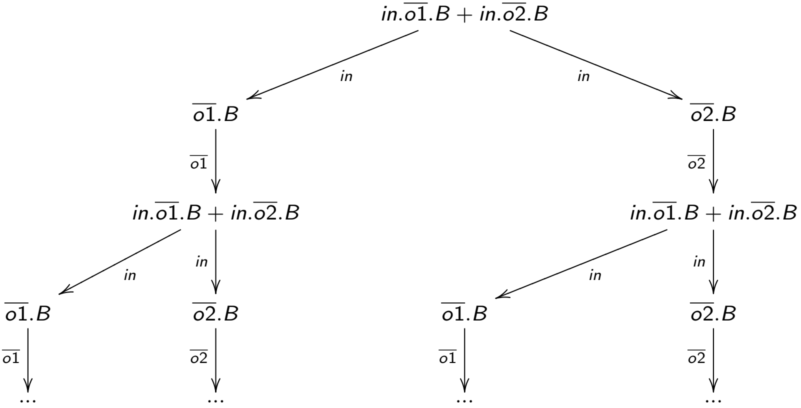
Transition tree

$B \triangleq in.\overline{o1}.B + in.\overline{o2}.B$



Transition graph

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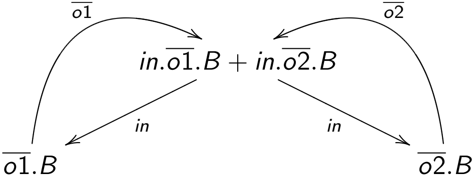

compare with $B' \triangleq in.(\overline{o1}.B' + \overline{o2}.B')$

Transition graph

$B \triangleq in.\overline{o1}.B + in.\overline{o2}.B$

$$
\begin{tikzcd}
& in.\overline{o1}.B + in.\overline{o2}.B \arrow[dl, "in"'] \arrow[dr, "in"] & \\
\overline{o1}.B \arrow[ur, bend left, "\overline{o1}"] & & \overline{o2}.B \arrow[ul, bend right, "\overline{o2}"']
\end{tikzcd}
$$

compare with $B' \triangleq in.(\overline{o1}.B' + \overline{o2}.B')$

$$
\begin{tikzcd}
in.(\overline{o1}.B' + \overline{o2}.B') \arrow[d, "in"'] \\
\overline{o1}.B' + \overline{o2}.B' \arrow[u, bend left=60, "\overline{o1}"] \arrow[u, bend right=60, "\overline{o2}"']
\end{tikzcd}
$$

Data parameters

Language $\mathbb{P}$ is extended to $\mathbb{P}_V$ over a data universe V, a set V_e of expressions over V and a evaluation $Val : V \longleftarrow V_e$

Example

$$B \triangleq in(x).B'_x$$

$$B'_v \triangleq \overline{out}\langle v\rangle.B$$

- Two prefix forms: $a(x).E$ and $\overline{a}\langle e\rangle.E$ (actions as ports)
- Data parameters: $A_S(x_1, ..., x_n) \triangleq E_A$, with $S \in V$ and each $x_i \in L$
- Conditional combinator: if b then P, if b then P_1 else P_2

Clearly

$$\text{if } b \text{ then } P_1 \text{ else } P_2 \stackrel{\text{abv}}{=} (\text{if } b \text{ then } P_1) + (\text{if } \neg b \text{ then } P_2)$$

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Data parameters

Additional semantic rules

$$\frac{}{\{v/x\}E \xleftarrow{a(v)} a(x).E}\ (\mathit{prefix}_i) \qquad \text{for } v \in V$$

$$\frac{}{E \xleftarrow{\overline{a}\langle v\rangle} \overline{a}\langle e\rangle.E}\ (\mathit{prefix}_o) \qquad \text{for } \mathit{Val}(e) = v$$

$$\frac{E' \xleftarrow{a} E_1}{E' \xleftarrow{a} \text{if } b \text{ then } E_1 \text{ else } E_2}\ (\mathit{if}_1) \qquad \text{for } \mathit{Val}(b) = \text{true}$$

$$\frac{E' \xleftarrow{a} E_2}{E' \xleftarrow{a} \text{if } b \text{ then } E_1 \text{ else } E_2}\ (\mathit{if}_2) \qquad \text{for } \mathit{Val}(b) = \text{false}$$

Back to PP

Encoding in the basic language: $\mathcal{T}(\ ) : \mathbb{P} \longleftarrow \mathbb{P}_V$

$$
\begin{aligned}
\mathcal{T}(a(x).E) &= \sum_{v \in V} a_v.\mathcal{T}(\{v/x\}E) \\
\mathcal{T}(\overline{a}\langle e \rangle.E) &= \overline{a}_e.\mathcal{T}(E) \\
\mathcal{T}(\sum_{i \in I} E_i) &= \sum_{i \in I} \mathcal{T}(E_i) \\
\mathcal{T}(E \mid F) &= \mathcal{T}(E) \mid \mathcal{T}(F) \\
\mathcal{T}(\text{new}\, K \;\, E) &= \text{new}\, \{a_v \mid a \in K, v \in V\}\; \mathcal{T}(E)
\end{aligned}
$$

and

$$
\mathcal{T}(\text{if}\, b\, \text{then}\, E) = \begin{cases} \mathcal{T}(E) & \text{if } Val(b) = \text{true} \\ \mathbf{0} & \text{if } Val(b) = \text{false} \end{cases}
$$

EX1: Canonical concurrent form

$$P \triangleq \text{new } K\ (E_1 \mid E_2 \mid \ldots \mid E_n)$$

The chance machine

$$\begin{aligned}
IO &\triangleq m.\overline{bank}.(lost.\overline{loss}.IO + rel(x).\overline{win}\langle x\rangle.IO) \\
B_n &\triangleq bank.\overline{max}\langle n+1\rangle.left(x).B_x \\
Dc &\triangleq max(z).(\overline{lost}.\overline{left}\langle z\rangle.Dc + \sum_{1\le x\le z} \overline{rel}\langle x\rangle.\overline{left}\langle z-x\rangle.Dc)
\end{aligned}$$

$$M_n \triangleq \text{new}\ \{bank, max, left, rel\}\ (IO \mid B_n \mid Dc)$$

EX2: Sequential patterns

1. List all states (configurations of variable assignments)
2. Define an order to capture systems's evolution
3. Specify an expression in $\mathbb{P}$ to define it

A 3-bit converter

$$
\begin{aligned}
A &\triangleq rq.B \\
B &\triangleq out0.C + out1.\overline{odd}.A \\
C &\triangleq out0.D + out1.\overline{even}.A \\
D &\triangleq out0.\overline{zero}.A + out1.\overline{even}.A
\end{aligned}
$$

EX3: The alternating-bit protocol

- protocol: set of rules orchestrating interaction between two entities to achieve a common goal
- ABP: exchange data over a unreliable medium: message loss and replication

EX3: ABP sender

- accepts message to deliver
- delivers message with bit b and sets a timer
- when a time-out in fired, re-sends b
- whenever a confirmation b is received, goes on with anew message and $1 - b$
- ignores any confirmation with $1 - b$

$$
\begin{aligned}
Accept_b &\triangleq accept \cdot Send_b \\[2em]
Send_b &\triangleq \overline{send}_b \cdot \overline{time} \cdot Sending_b \\
Sending_b &\triangleq timeout \cdot Send_b + ack_b \cdot timeout \cdot Accept_{1-b} \\
&\quad + ack_{1-b} \cdot Sending_b
\end{aligned}
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EX3: ABP receiver

- receives a message and delivers it its client
- sends confirmation with bit b and sets a timer
- when a time-out in fired, re-sends b
- whenever receives a new message with $1 - b$, delivers it its client, and continues with $1 - b$
- ignores any message with b

$$Deliver_b \triangleq deliver \cdot Reply_b$$

$$\begin{aligned} Reply_b &\triangleq \overline{reply}_b \cdot \overline{time} \cdot Replying_b \\ Replying_b &\triangleq timeout \cdot Reply_b + trans_{1-b} \cdot timeout \cdot Deliver_{1-b} \\ &\quad + trans_b \cdot Replying_b \end{aligned}$$

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EX3: ABP composing with timers

$$
\begin{aligned}
\mathit{Timer} &\triangleq \mathit{time} \cdot \overline{\mathit{timeout}} \cdot \mathit{Timer} \\
\mathit{Sender}_b &\triangleq \mathit{accept}.\mathrm{new}\,\{\mathit{time}, \mathit{timeout}\}\,(\mathit{Send}_b \mid \mathit{Timer}) \\
\mathit{Receiver}_b &\triangleq \mathrm{new}\,\{\mathit{time}, \mathit{timeout}\}\,(\mathit{Reply}_b \mid \mathit{Timer})
\end{aligned}
$$

EX3: ABP communication medium

$$
\begin{aligned}
Trans_{sb} &\triangleq \overline{trans}_b \cdot Trans_s \\
Trans_s &\triangleq send_b \cdot Trans_{bs} \\
Trans_{tbs} &\triangleq \tau \cdot Trans_{ts} \\
Trans_{tbs} &\triangleq \tau \cdot Trans_{tbbs}
\end{aligned}
$$

and

$$
\begin{aligned}
Ack_{bs} &\triangleq \overline{ack}_b \cdot Ack_s \\
Ack_s &\triangleq reply_b \cdot Ack_{sb} \\
Ack_{sbt} &\triangleq \tau \cdot Ack_{st} \\
Ack_{sbt} &\triangleq \tau \cdot Ack_{sbbt}
\end{aligned}
$$

EX3: ABP - the protocol

$$\boxed{AB \triangleq \text{new } K\ (Sender_{1-b} \mid Trans_{\epsilon} \mid Ack_{\epsilon} \mid Receiver_{b})}$$

where $K \ = \ \{send_b, ack_b, reply_b, trans_b \mid b \in \{0,1\}\}$.

Processes are 'prototypical' transition systems

... hence all definitions apply:

$E \sim F$

- Processes E, F are bisimilar if there exist a bisimulation S st $\{\langle E, F\rangle\} \in S$.
- A binary relation S in $\mathbb{P}$ is a (strict) bisimulation iff, whenever $(E, F) \in S$ and $a \in Act$,

$$
\begin{array}{rl}
\text{i)} & E' \xleftarrow{a} E \;\Rightarrow\; F' \xleftarrow{a} F \;\wedge\; (E', F') \in S \\
\text{ii)} & F' \xleftarrow{a} F \;\Rightarrow\; E' \xleftarrow{a} E \;\wedge\; (E', F') \in S
\end{array}
$$

I.e.,

$$\sim \;=\; \bigcup \{S \subseteq \mathbb{P} \times \mathbb{P} \mid S \quad \text{is a (strict) bisimulation}\}$$

Processes are 'prototipycal' transition systems

Example: $S \sim M$

$$T \triangleq i.\overline{k}.T$$
$$R \triangleq k.j.R$$
$$S \triangleq \text{new}\,\{k\}\;(T \mid R)$$

$$M \triangleq i.\tau.N$$
$$N \triangleq j.i.\tau.N + i.j.\tau.N$$

through bisimulation

$$R = \{\langle S, M)\rangle, \langle \text{new}\,\{k\}\;(\overline{k}.T \mid R), \tau.N\rangle, \langle \text{new}\,\{k\}\;(T \mid j.R), N\rangle, \langle \text{new}\,\{k\}\;(\overline{k}.T \mid j.R), j.\tau.N\rangle\}$$

Example: Semaphores

A semaphore

$$Sem \triangleq get.put.Sem$$

n-semaphores

$$\begin{aligned} Sem_n &\triangleq Sem_{n,0} \\ Sem_{n,0} &\triangleq get.Sem_{n,1} \\ Sem_{n,i} &\triangleq get.Sem_{n,i+1} + put.Sem_{n,i-1} \\ &\quad (\text{for } 0 < i < n) \\ Sem_{n,n} &\triangleq put.Sem_{n,n-1} \end{aligned}$$

Sem_n can also be implemented by the parallel composition of n Sem processes:

$$Sem^n \triangleq Sem \mid Sem \mid \ldots \mid Sem$$

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Example: Semaphores

Is $Sem_n \;\sim\; Sem^n$?

For $n = 2$:

$$\{\langle Sem_{2,0}, Sem \mid Sem\rangle, \langle Sem_{2,1}, Sem \mid put.Sem\rangle,$$
$$\langle Sem_{2,1}, put.Sem \mid Sem\rangle\langle Sem_{2,2}, put.Sem \mid put.Sem\rangle\}$$

is a bisimulation.

- but can we get rid of structurally congruent pairs?

Example: Semaphores

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Bisimulation up to $\equiv$

Definition

A binary relation S in $\mathbb{P}$ is a (strict) bisimulation up to $\equiv$ iff, whenever $(E, F) \in S$ and $a \in Act$,

$$\text{i)}\quad E' \xleftarrow{a} E \;\Rightarrow\; F' \xleftarrow{a} F \wedge (E', F') \in \equiv \cdot S \cdot \equiv$$

$$\text{ii)}\quad F' \xleftarrow{a} F \;\Rightarrow\; E' \xleftarrow{a} E \wedge (E', F') \in \equiv \cdot S \cdot \equiv$$

Lemma

If S is a (strict) bisimulation up to $\equiv$, then $S \subseteq\ \sim$

- To prove $Sem_n \sim Sem^n$ a bisimulation will contain 2^n pairs, while a bisimulation up to $\equiv$ only requires $n+1$ pairs.

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A $\sim$-calculus

Lemma

$$E \equiv F \;\Rightarrow\; E \sim F$$

- proof idea: show that $\{(E + E, E) \mid E \in \mathbb{P}\} \cup Id_{\mathbb{P}}$ is a bisimulation

Lemma

$$\text{new}\, K' \;(\text{new}\, K\;\; E) \sim \text{new}\,(K \cup K')\;E$$

$$\text{new}\, K\;\; E \sim E \qquad \text{if } \mathbb{L}(E) \cap (K \cup \overline{K}) = \emptyset$$

$$\text{new}\, K\;(E \mid F) \sim \text{new}\, K\;\; E \mid \text{new}\, K\;\; F \qquad \text{if } \mathbb{L}(E) \cap \overline{\mathbb{L}(F)} \cap (K \cup \overline{K}) = \emptyset$$

- proof idea: discuss whether S is a bisimulation:

$$S \;=\; \{(\text{new}\, K\;\; E, E) \mid E \in \mathbb{P} \wedge \mathbb{L}(E) \cap (K \cup \overline{K}) = \emptyset\}$$

A $\sim$-calculus

Lemma

$$E \equiv F \;\Rightarrow\; E \sim F$$

- proof idea: show that $\{(E + E, E) \mid E \in \mathbb{P}\} \cup Id_{\mathbb{P}}$ is a bisimulation

Lemma

$$\text{new}\, K' \,(\text{new}\, K \;\, E) \sim \text{new}\,(K \cup K')\; E$$

$$\text{new}\, K \;\, E \sim E \qquad \text{if } \mathbb{L}(E) \cap (K \cup \overline{K}) = \emptyset$$

$$\text{new}\, K \;(E \mid F) \sim \text{new}\, K \;\, E \mid \text{new}\, K \;\, F \qquad \text{if } \mathbb{L}(E) \cap \overline{\mathbb{L}(F)} \cap (K \cup \overline{K}) = \emptyset$$

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$\sim$ is a congruence

$\boxed{\text{congruence is the name of modularity in Mathematics}}$

- process combinators preserve $\sim$

Lemma

$$
\begin{aligned}
a.E &\sim a.F \\
E + P &\sim F + P \\
E \mid P &\sim F \mid P \\
\text{new}\ K\ E &\sim \text{new}\ K\ F
\end{aligned}
$$

- recursive definition preserves $\sim$

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E \mid P \sim F \mid P \\
\text{new } K \ E \sim \text{new } K \ F
\end{gathered}
$$

- recursive definition preserves $\sim$

$\sim$ is a congruence

- First $\sim$ is extended to processes with variables:

$$E \sim F \;\Leftrightarrow\; \forall_{\tilde{P}}\,.\;\{\tilde{P}/\tilde{X}\}\,E \sim \{\tilde{P}/\tilde{X}\}\,F$$

- Then prove:

Lemma

i) $\tilde{P} \triangleq \tilde{E} \;\Rightarrow\; \tilde{P} \sim \tilde{E}$
where $\tilde{E}$ is a family of process expressions and $\tilde{P}$ a family of process identifiers.

ii) Let $\tilde{E} \sim \tilde{F}$, where $\tilde{E}$ and $\tilde{F}$ are families of recursive process expressions over a family of process variables $\tilde{X}$, and define:

$$\tilde{A} \triangleq \{\tilde{A}/\tilde{X}\}\,\tilde{E} \;\text{ and }\; \tilde{B} \triangleq \{\tilde{B}/\tilde{X}\}\,\tilde{F}$$

Then

$$\tilde{A} \sim \tilde{B}$$

The expansion theorem

$\boxed{\text{Every process is equivalent to the sum of its derivatives}}$

$$E \sim \sum \{a.E' \mid E' \xleftarrow{a} E\}$$

understood?

$$E \sim \sum \{a.E' \mid E' \xleftarrow{a} E\}$$

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clear?

$$E \sim \sum \{a.E' \mid E' \xleftarrow{a} E\}$$

The expansion theorem

The usual definition (based on the concurrent canonical form):

$$
\begin{aligned}
E \sim {} & \sum \{\, f_i(a).\mathsf{new}\, K\ (\{f_1\}\, E_1 \mid \ldots \mid \{f_i\}\, E_i' \mid \ldots \mid \{f_n\}\, E_n) \ \mid \\
& \qquad\qquad\qquad E_i' \xleftarrow{\ a\ } E_i \ \wedge\ f_i(a) \notin K \cup \overline{K} \,\} \\
& + \\
& \sum \{\, \tau.\mathsf{new}\, K\ (\{f_1\}\, E_1 \mid \ldots \mid \{f_i\}\, E_i' \mid \ldots \mid \{f_j\}\, E_j' \mid \ldots \mid \{f_n\}\, E_n) \ \mid \\
& \qquad\qquad\qquad E_i' \xleftarrow{\ a\ } E_i \ \wedge\ E_j' \xleftarrow{\ b\ } E_j \ \wedge\ f_i(a) = \overline{f_j(b)} \,\}
\end{aligned}
$$

for $E \triangleq \mathsf{new}\, K\ (\{f_1\}\, E_1 \mid \ldots \mid \{f_n\}\, E_n)$, with $n \geq 1$

The expansion theorem

Corollary (for $n = 1$ and $f_1 = \mathrm{id}$)

$$\mathrm{new}\,K\ (E + F) \sim \mathrm{new}\,K\ E + \mathrm{new}\,K\ F$$

$$\mathrm{new}\,K\ (a.E) \sim \begin{cases} \mathbf{0} & \text{if } a \in (K \cup \overline{K}) \\ a.(\mathrm{new}\,K\ E) & \text{otherwise} \end{cases}$$

Example

$S \sim M$

$$\begin{aligned}
S &\sim \text{new}\,\{k\}\;(T \mid R)\\
&\sim i.\text{new}\,\{k\}\;(\overline{k}.T \mid R)\\
&\sim i.\tau.\text{new}\,\{k\}\;(T \mid j.R)\\
&\sim i.\tau.(i.\text{new}\,\{k\}\;(\overline{k}.T \mid j.R) + j.\text{new}\,\{k\}\;(T \mid R))\\
&\sim i.\tau.(i.j.\text{new}\,\{k\}\;(\overline{k}.T \mid R) + j.i.\text{new}\,\{k\}\;(\overline{k}.T \mid R))\\
&\sim i.\tau.(i.j.\tau.\text{new}\,\{k\}\;(T \mid j.R) + j.i.\tau.\text{new}\,\{k\}\;(T \mid j.R))
\end{aligned}$$

Let $N' = \text{new}\,\{k\}\;(T \mid j.R)$.
This expands into $N' \sim i.j.\tau.\text{new}\,\{k\}\;(T \mid j.R) + j.i.\tau.\text{new}\,\{k\}\;(T \mid j.R)$,
Therefore $N' \sim N$ and $S \sim i.\tau.N \sim M$

- requires result on unique solutions for recursive process equations

Observable transitions

$$\overset{a}{\Longleftarrow} \subseteq \mathbb{P} \times \mathbb{P}$$

- $L \cup \{\epsilon\}$
- A $\overset{\epsilon}{\Longleftarrow}$-transition corresponds to zero or more non observable transitions
- inference rules for $\overset{a}{\Longleftarrow}$:

$$\frac{}{E \overset{\epsilon}{\Longleftarrow} E}\ (O_1)$$

$$\frac{E \overset{\tau}{\longleftarrow} E' \quad E' \overset{\epsilon}{\Longleftarrow} F}{E \overset{\epsilon}{\Longleftarrow} F}\ (O_2)$$

$$\frac{E \overset{\epsilon}{\Longleftarrow} E' \quad E' \overset{a}{\longleftarrow} F' \quad F' \overset{\epsilon}{\Longleftarrow} F}{E \overset{a}{\Longleftarrow} F}\ (O_3) \quad \text{for } a \in L$$

Example

$$
\begin{aligned}
T_0 &\triangleq j.T_1 + i.T_2 \\
T_1 &\triangleq i.T_3 \\
T_2 &\triangleq j.T_3 \\
T_3 &\triangleq \tau.T_0
\end{aligned}
$$

and

$$A \triangleq i.j.A + j.i.A$$

Example

From their graphs,

$$\begin{tikzcd}
 & T_0 \arrow[dl, "j"'] \arrow[dr, "i"] & \\
T_1 \arrow[dr, "i"'] & & T_2 \arrow[dl, "j"] \\
 & T_3 \arrow[uu, bend right=90, "\tau"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & A \arrow[dl, "j"] \arrow[dr, "i"'] & \\
i.A \arrow[ur, bend left=60, "i"] & & j.A \arrow[ul, bend right=60, "j"']
\end{tikzcd}$$

we conclude that $T_0 \not\approx A$ (why?).

Observational equivalence

$E \approx F$

- Processes E, F are observationally equivalent if there exists a weak bisimulation S st $\{\langle E, F\rangle\} \in S$.
- A binary relation S in $\mathbb{P}$ is a weak bisimulation iff, whenever $(E, F) \in S$ and $a \in L \cup \{\epsilon\}$,

$$\begin{array}{rl} \text{i)} & E' \stackrel{a}{\Longleftarrow} E \;\Rightarrow\; F' \stackrel{a}{\Longleftarrow} F \;\wedge\; (E', F') \in S \\ \text{ii)} & F' \stackrel{a}{\Longleftarrow} F \;\Rightarrow\; E' \stackrel{a}{\Longleftarrow} E \;\wedge\; (E', F') \in S \end{array}$$

I.e.,

$$\approx \;=\; \bigcup \{S \subseteq \mathbb{P} \times \mathbb{P} \mid S \text{ is a weak bisimulation}\}$$

Observational equivalence

Properties

- as expected: $\approx$ is an equivalence relation
- basic property: for any $E \in \mathbb{P}$,

$$E \;\approx\; \tau.E$$

(proof idea: $\mathrm{id}_{\mathbb{P}} \cup \{(E, \tau.E) \mid E \in \mathbb{P}\}$ is a weak bisimulation

- weak vs. strict:

$$\sim \subseteq \approx$$

Is $\approx$ a congruence?

Lemma

Let $E \approx F$. Then, for any $P \in \mathbb{P}$ and $K \subseteq L$,

$$a.E \approx a.F$$
$$E \mid P \approx F \mid P$$
$$\text{new}\, K \;\, E \approx \text{new}\, K \;\, F$$

but

$$E + P \approx F + P$$

does not hold, in general.

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Example (initial τ restricts options 'menu')

$$i.\mathbf{0} \;\approx\; \tau.i.\mathbf{0}$$

However

$$j.\mathbf{0} + i.\mathbf{0} \not\approx j.\mathbf{0} + \tau.i.\mathbf{0}$$

Actually,

$$\begin{tikzcd}
 & j.\mathbf{0} + i.\mathbf{0} \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathbf{0} & & \mathbf{0}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & j.\mathbf{0} + \tau.i.\mathbf{0} \arrow[dl, "j"'] \arrow[dr, "\tau"] & \\
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\end{tikzcd}$$

Forcing a congruence: $E = F$

Solution: force any initial τ to be matched by another τ

Process equality

Two processes E and F are equal (or observationally congruent) iff

i) $E \approx F$

ii) $E' \xleftarrow{\tau} E \;\Rightarrow\; F' \xLeftarrow{\epsilon} F'' \xleftarrow{\tau} F$ and $E' \approx F'$

iii) $F' \xleftarrow{\tau} F \;\Rightarrow\; E' \xLeftarrow{\epsilon} E'' \xleftarrow{\tau} E$ and $E' \approx F'$

- note that $E \neq \tau.E$, but $\tau.E = \tau.\tau.E$

Forcing a congruence: $E = F$

$\boxed{\text{Solution: force any initial } \tau \text{ to be matched by another } \tau}$

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Forcing a congruence: $E = F$

$=$ can be regarded as a restriction of $\approx$ to all pairs of processes which preserve it in additive contexts

Lemma

Let E and F be processes such that the union of their sorts is distinct of L.

$$E = F \;\Leftrightarrow\; \forall_{G \in \mathbb{P}} \,.\, (E + G \approx F + G)$$

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Properties of $=$

Lemma

$$E = F \;\Rightarrow\; \forall_{G \in \mathbb{P}} \,.\, (E + G \approx F + G)$$

Lemma

$$E = F \;\Leftrightarrow\; (E = F) \vee (E = \tau.F) \vee (\tau.E = F)$$

Properties of $=$

Lemma

$$\sim \; \subseteq = \subseteq \; \approx$$

So,

$$\boxed{\text{the whole} \sim \text{theory remains valid}}$$

Additionally,

Lemma (additional laws)

$$a.\tau.E = a.E$$
$$E + \tau.E = \tau.E$$
$$a.(E + \tau.F) = a.(E + \tau.F) + a.F$$

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Solving equations

Have equations over $(\mathbb{P}, \sim)$ or $(\mathbb{P}, =)$ (unique) solutions?

Lemma

Recursive equations $\tilde{X} = \tilde{E}(\tilde{X})$ or $\tilde{X} \sim \tilde{E}(\tilde{X})$, over $\mathbb{P}$, have unique solutions (up to $=$ or $\sim$, respectively). Formally,

i) Let $\tilde{E} = \{E_i \mid i \in I\}$ be a family of expressions with a maximum of I free variables ($\{X_i \mid i \in I\}$) such that any variable free in E_i is weakly guarded. Then

$$\tilde{P} \sim \{\tilde{P}/\tilde{X}\}\tilde{E} \wedge \tilde{Q} \sim \{\tilde{Q}/\tilde{X}\}\tilde{E} \Rightarrow \tilde{P} \sim \tilde{Q}$$

ii) Let $\tilde{E} = \{E_i \mid i \in I\}$ be a family of expressions with a maximum of I free variables ($\{X_i \mid i \in I\}$) such that any variable free in E_i is guarded and sequential. Then

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Conditions on variables

guarded :
X occurs in a sub-expression of type $a.E'$ for $a \in Act - \{\tau\}$

weakly guarded :
X occurs in a sub-expression of type $a.E'$ for $a \in Act$

in both cases assures that, until a guard is reached, behaviour does not depends on the process that instantiates the variable

example: X is weakly guarded in both $\tau.X$ and $\tau.\mathbf{0} + a.X + b.a.X$ but guarded only in the second

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Conditions on variables

sequential :

X is sequential in E if every strict sub-expression in which X occurs is either $a.E'$, for $a \in Act$, or $\Sigma\tilde{E}$.

$\boxed{\text{avoids } X \text{ to become guarded by a } \tau \text{ as a result of an interaction}}$

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example: X is not sequential in $X = \text{new}\,\{a\}\;(\overline{a}.X \mid a.\mathbf{0})$

Example (1)

Consider

$$\begin{aligned}
Sem &\triangleq get.put.Sem \\
P_1 &\triangleq \overline{get}.c_1.\overline{put}.P_1 \\
P_2 &\triangleq \overline{get}.c_2.\overline{put}.P_2 \\
S &\triangleq \text{new}\,\{get, put\}\ (Sem \mid P_1 \mid P_2)
\end{aligned}$$

and

$$S' \triangleq \tau.c_1.S' + \tau.c_2.S'$$

to prove $S \sim S'$, show both are solutions of

$$X = \tau.c_1.X + \tau.c_2.X$$

Example (1)

Consider

$$
\begin{aligned}
Sem &\triangleq get.put.Sem \\
P_1 &\triangleq \overline{get}.c_1.\overline{put}.P_1 \\
P_2 &\triangleq \overline{get}.c_2.\overline{put}.P_2 \\
S &\triangleq \text{new}\,\{get, put\}\ (Sem \mid P_1 \mid P_2)
\end{aligned}
$$

and

$$S' \triangleq \tau.c_1.S' + \tau.c_2.S'$$

to prove $S \sim S'$, show both are solutions of

$$X \;=\; \tau.c_1.X + \tau.c_2.X$$

Example (1)

proof

$$
\begin{aligned}
S = \; & \tau.\text{new}\,K\;(c_1.\overline{put}.P_1 \mid P_2 \mid put.Sem) + \tau.\text{new}\,K\;(P_1 \mid c_2.\overline{put}.P_2 \mid put.Sem) \\
= \; & \tau.c_1.\text{new}\,K\;(\overline{put}.P_1 \mid P_2 \mid put.Sem) + \tau.c_2.\text{new}\,K\;(P_1 \mid \overline{put}.P_2 \mid put.Sem) \\
= \; & \tau.c_1.\tau.\text{new}\,K\;(P_1 \mid P_2 \mid Sem) + \tau.c_2.\tau.\text{new}\,K\;(P_1 \mid P_2 \mid Sem) \\
= \; & \tau.c_1.\tau.S + \tau.c_2.\tau.S \\
= \; & \tau.c_1.S + \tau.c_2.S \\
= \; & \{S/X\}E
\end{aligned}
$$

for S' is immediate

Example (2)

Consider,

$$B \triangleq in.B_1 \qquad B' \triangleq \text{new } m\ (C_1 \mid C_2)$$
$$B_1 \triangleq in.B_2 + \overline{out}.B \qquad C_1 \triangleq in.\overline{m}.C_1$$
$$C_2 \triangleq m.\overline{out}.C_2$$

B' is a solution of

$$X = E(X, Y, Z) = in.Y$$
$$Y = E_1(X, Y, Z) = in.Z + \overline{out}.X$$
$$Z = E_3(X, Y, Z) = \overline{out}.Y$$

through $\sigma = \{B/X, B_1/Y, B_2/Z\}$

Example (2)

To prove B =B'

$$
\begin{aligned}
B' &= \text{new } m \; (C_1 \mid C_2) \\
&= in.\text{new } m \; (\overline{m}.C_1 \mid C_2) \\
&= in.\tau.\text{new } m \; (C_1 \mid \overline{out}.C_2) \\
&= in.\text{new } m \; (C_1 \mid \overline{out}.C_2)
\end{aligned}
$$

Let $S_1 = \text{new } m \; (C_1 \mid \overline{out}.C_2)$ to proceed:

$$
\begin{aligned}
S_1 &= \text{new } m \; (C_1 \mid \overline{out}.C_2) \\
&= in.\text{new } m \; (\overline{m}.C_1 \mid \overline{out}.C_2) + \overline{out}.\text{new } m \; (C_1 \mid C_2) \\
&= in.\text{new } m \; (\overline{m}.C_1 \mid \overline{out}.C_2) + \overline{out}.B'
\end{aligned}
$$

Example (2)

Finally, let, $S_2 = \text{new}\, m\, (\overline{m}.C_1 \mid \overline{out}.C_2)$. Then,

$$
\begin{aligned}
S_2 &= \text{new}\, m\, (\overline{m}.C_1 \mid \overline{out}.C_2) \\
&= \overline{out}.\text{new}\, m\, (\overline{m}.C_1 \mid C_2) \\
&= \overline{out}.\tau.\text{new}\, m\, (C_1 \mid \overline{out}.C_2) \\
&= \overline{out}.\tau.S_1 \\
&= \overline{out}.S_1
\end{aligned}
$$

Example (2)

Note the same problem can be solved with a system of 2 equations:

$$
\begin{aligned}
X = &\ E(X, Y) = in.Y \\
Y = &\ E'(X, Y) = in.\overline{out}.Y + \overline{out}.in.Y
\end{aligned}
$$

Clearly, by substitution,

$$
\begin{aligned}
B = &\ in.B_1 \\
B_1 = &\ in.\overline{out}.B_1 + \overline{out}.in.B_1
\end{aligned}
$$

Example (2)

On the other hand, it's already proved that $B' = ... = in.S_1$.
so,

$$\begin{aligned}
S_1 &= \text{new}\, m\, (C_1 \mid \overline{out}.C_2) \\
&= in.\text{new}\, m\, (\overline{m}.C_1 \mid \overline{out}.C_2) + \overline{out}.B' \\
&= in.\overline{out}.\text{new}\, m\, (\overline{m}.C_1 \mid C_2) + \overline{out}.B' \\
&= in.\overline{out}.\tau.\text{new}\, m\, (C_1 \mid \overline{out}.C_2) + \overline{out}.B' \\
&= in.\overline{out}.\tau.S_1 + \overline{out}.B' \\
&= in.\overline{out}.S_1 + \overline{out}.B' \\
&= in.\overline{out}.S_1 + \overline{out}.in.S_1
\end{aligned}$$

Hence, $B' = \{B'/X, S_1/Y\}E$ and $S_1 = \{B'/X, S_1/Y\}E'$