Labelled Transition Systems (I)

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March, 2010

Reactive systems

Reactive system

system that computes by reacting to stimuli from its environment along its overall computation

- in contrast to sequential systems whose meaning is defined by the results of finite computations, the behaviour of reactive systems is mainly determined by interaction and mobility of non-terminating processes, evolving concurrently.
- observation $\Leftrightarrow$ interaction
- behaviour $\Leftrightarrow$ a structured record of interactions

Reactive systems

Concurrency vs interaction

$$x := 0;$$
$$x := x + 1 \mid x := x + 2$$

- both statements in parallel could read x before it is written
- which values can x take?
- which is the program outcome if exclusive access to memory and atomic execution of assignments is guaranteed?

Models of computation for continuous interaction

two reactive systems you are already familiar with

Functions $f : O \longleftarrow I$

- one-step, input-output behaviour
- but what about functions manipulating infinite data structures?

$$\text{merge} : A^\omega \longleftarrow A^\omega \times A^\omega$$

Automata

- multi-step behaviour: accepted language

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Automata

- multi-step behaviour: accepted language

Ex: Functions over streams

Streams are coalgebraic structures: specified by observers

$$\langle \mathsf{hd}, \mathsf{tl} \rangle : A \times A^{\omega} \longleftarrow A^{\omega}$$

- Function $\langle \mathsf{hd}, \mathsf{tl} \rangle$ is the observation structure of A^{ω}.
- The shape of such an observation is given by functor $\mathsf{T} : A \times X \longleftarrow X$ for which $\langle \mathsf{hd}, \mathsf{tl} \rangle$ is a coalgebra.

Coalgebra

a lens:	$\bigcirc\!\frown\!\bigcirc$
a tool box:	$\overset{\Cap}{\square\!\square}$
an observation structure:	$\bigcirc\!\frown\!\bigcirc$ universe $\xleftarrow{\alpha}$ universe
an assembly process:	artifact $\xleftarrow{d}$ $\overset{\Cap}{\square\!\square}$ artifact

$$\alpha : \mathsf{F}\, U \longleftarrow U$$

- coalgebras describe transition systems
- and abstract behaviour types as (final) coalgebras
- compare with (initial) algebras and (finite) data structures

Coalgebra

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Ex: Functions over streams

- Coalgebras

$$p \;=\; \langle \mathsf{at}, \mathsf{m} \rangle : A \times U \longleftarrow U$$

for the same functor, relate through morphisms:
structure-preserving functions,

$$\begin{array}{ccc} U & \xrightarrow{\langle \mathsf{at},\mathsf{m} \rangle} & A \times U \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathsf{id} \times h} \\ V & \xrightarrow{\langle \mathsf{at}',\mathsf{m}' \rangle} & A \times V \end{array}$$

$$\mathsf{at} = \mathsf{at}' \cdot h \quad \text{and} \quad h \cdot \mathsf{m} = \mathsf{m}' \cdot h$$

- The behaviour of $\langle \mathsf{at}, \mathsf{m} \rangle$, from an initial value u, is given by successive observations:

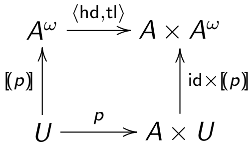
$$[\![p]\!]\; u \;=\; [\mathsf{at}\; u, \mathsf{at}\; (\mathsf{m}\; u), \mathsf{at}\; (\mathsf{m}\; (\mathsf{m}\; u)), \ldots]$$

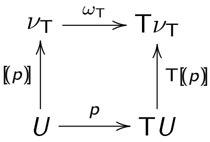
originating a stream of A values.

Ex: Functions over streams

$$\langle \mathsf{hd}, \mathsf{tl} \rangle : A \times A^\omega \longleftarrow A^\omega$$

is final, i.e. characterised by the following universal property: from any other coalgebra p there is a unique morphism $[\![p]\!]$ st





$$k = [\![p]\!] \quad \Leftrightarrow \quad \omega_\mathsf{T} \cdot k = \mathsf{T}\ k \cdot p$$

from where one derives the usual toolkit:

cancelation	$\omega_\mathsf{T} \cdot [\![p]\!] = \mathsf{T}\ [\![p]\!] \cdot p$
reflection	$[\![\omega_\mathsf{T}]\!] = \mathsf{id}_{\nu_\mathsf{T}}$
fusion	$[\![p]\!] \cdot h = [\![q]\!]$ if $p \cdot h = \mathsf{T}\ h \cdot q$

Ex: Functions over streams

$$\langle \mathsf{hd}, \mathsf{tl} \rangle : A \times A^{\omega} \longleftarrow A^{\omega}$$

is final, i.e. characterised by the following universal property: from any other coalgebra p there is a unique morphism $\llbracket p \rrbracket$ st

$$\begin{array}{ccc} A^{\omega} & \xrightarrow{\langle \mathsf{hd},\mathsf{tl} \rangle} & A \times A^{\omega} \\ {\scriptstyle \llbracket p \rrbracket}\uparrow & & \uparrow{\scriptstyle \mathsf{id} \times \llbracket p \rrbracket} \\ U & \xrightarrow{p} & A \times U \end{array} \qquad \begin{array}{ccc} \nu_{\mathsf{T}} & \xrightarrow{\omega_{\mathsf{T}}} & \mathsf{T}\nu_{\mathsf{T}} \\ {\scriptstyle \llbracket p \rrbracket}\uparrow & & \uparrow{\scriptstyle \mathsf{T}\llbracket p \rrbracket} \\ U & \xrightarrow{p} & \mathsf{T}U \end{array}$$

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Ex: Functions over streams

Behaviour is specified under all observers

Example:

$$
\begin{array}{ccc}
A^{\omega} & \xrightarrow{\langle \mathsf{hd}, \mathsf{tl} \rangle} & A \times A^{\omega} \\
\uparrow{\scriptstyle \mathsf{rep}} & & \uparrow{\scriptstyle \mathsf{id} \times \mathsf{rep}} \\
A & \xrightarrow{\triangle} & A \times A
\end{array}
$$

$$\mathsf{rep} \ \triangleq \ [\![(\triangle)]\!]$$

Definition by coinduction

$$
\begin{aligned}
& (\mathsf{id} \times \mathsf{rep}) \cdot \triangle = \langle \mathsf{hd}, \mathsf{tl} \rangle \cdot \mathsf{rep} \\
\Leftrightarrow \quad & \{ \ \triangle \text{ definition } \} \\
& (\mathsf{id} \times \mathsf{rep}) \cdot \langle \mathsf{id}, \mathsf{id} \rangle = \langle \mathsf{hd}, \mathsf{tl} \rangle \cdot \mathsf{rep} \\
\Leftrightarrow \quad & \{ \ \times \text{ abs and fusion } \} \\
& \langle \mathsf{id}, \mathsf{rep} \rangle = \langle \mathsf{hd} \cdot \mathsf{rep}, \mathsf{tl} \cdot \mathsf{rep} \rangle \\
\Leftrightarrow \quad & \{ \text{ structural equality } \} \\
& \mathsf{hd} \cdot \mathsf{rep} = \mathsf{id} \ \wedge \ \mathsf{tl} \cdot \mathsf{rep} = \mathsf{rep} \\
\Leftrightarrow \quad & \{ \text{ going pointwise } \} \\
& \mathsf{hd}\ (\mathsf{rep}\ a) = a \ \wedge \ \mathsf{tl}\ (\mathsf{rep}\ a) = \mathsf{rep}\ a
\end{aligned}
$$

Exercise: define merge and twist.

Proof by coinduction: $\mathsf{merge}\,(a^\omega, b^\omega) \;=\; (ab)^\omega$

$$
\begin{array}{ll}
 & \mathsf{merge} \cdot (\mathsf{rep} \times \mathsf{rep}) \;=\; \mathsf{twist} \\
= & \qquad \{\;\text{merge definition}\;\} \\
 & [\![\langle \mathsf{hd} \cdot \pi_1, \mathsf{s} \cdot (\mathsf{tl} \times \mathsf{id})\rangle]\!] \cdot (\mathsf{rep} \times \mathsf{rep}) \;=\; [\![\langle \pi_1, \mathsf{s}\rangle]\!] \\
\Leftarrow & \qquad \{\;\text{fusion}\;\} \\
 & \langle \mathsf{hd} \cdot \pi_1, \mathsf{s} \cdot (\mathsf{tl} \times \mathsf{id})\rangle \cdot (\mathsf{rep} \times \mathsf{rep}) \;=\; \mathsf{id} \times (\mathsf{rep} \times \mathsf{rep}) \cdot \langle \pi_1, \mathsf{s}\rangle \\
= & \qquad \{\;\times\text{ abs and reflection}\;\} \\
 & \langle \mathsf{hd} \cdot \mathsf{rep} \cdot \pi_1, \mathsf{s} \cdot ((\mathsf{tl} \cdot \mathsf{rep}) \times \mathsf{rep})\rangle \;=\; \mathsf{id} \times (\mathsf{rep} \times \mathsf{rep}) \cdot \langle \pi_1, \mathsf{s}\rangle \\
= & \qquad \{\;\mathsf{tl} \cdot \mathsf{rep} = \mathsf{rep} \text{ e } \mathsf{hd} \cdot \mathsf{rep} = \mathsf{id}\;\} \\
 & \langle \pi_1, \mathsf{s} \cdot (\mathsf{rep} \times \mathsf{rep})\rangle \;=\; \mathsf{id} \times (\mathsf{rep} \times \mathsf{rep}) \cdot \langle \pi_1, \mathsf{s}\rangle \\
= & \qquad \{\;\times\text{ abs}\;\} \\
 & \langle \pi_1, \mathsf{s} \cdot (\mathsf{rep} \times \mathsf{rep})\rangle \;=\; \langle \pi_1, (\mathsf{rep} \times \mathsf{rep}) \cdot \mathsf{s}\rangle \\
= & \qquad \{\;\mathsf{s}\text{ natural: } (f \times g) \cdot \mathsf{s} = \mathsf{s} \cdot (g \times f)\;\} \\
 & \langle \pi_1, \mathsf{s} \cdot (\mathsf{rep} \times \mathsf{rep})\rangle \;=\; \langle \pi_1, \mathsf{s} \cdot (\mathsf{rep} \times \mathsf{rep})\rangle
\end{array}
$$

Ex: Automata

Definition

$A = \langle \Sigma, S, s_o, F, T \rangle$
where

- Σ is an alphabet
- $S = \{s_0, s_1, s_2, ...\}$ is a set of states
- $s_0 \in S$ is the initial state
- $F \subseteq S$ is the set of final states
- $T \subseteq S \times \Sigma \times S$ is the transition relation usually given as a Σ-indexed family of realtions over S:

$$s \xrightarrow{a} s' \;\Leftrightarrow\; \langle s', a, s \rangle \in T$$

- deterministic
- finite
- image finite

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Ex: Automata

automaton behaviour $\Leftrightarrow$ accepted language

Recall that finite automata recognize regular languages, i.e. generated by

- $L_1 + L_2 \triangleq L_1 \cup L_2$ (union)
- $L_1 \cdot L_2 \triangleq \{st \mid s \in L_1, t \in L_2\}$ (concatenation)
- $L^* \triangleq \{\epsilon\} \cup L \cup (L \cdot L) \cup (L \cdot L \cdot L) \cup \ldots$ (iteration)

Ex: Automata

There is a syntax to specify such languages:

$$E \;::=\; \epsilon \mid a \mid E + E \mid E\,E \mid E^*$$

where $a \in \Sigma$.

- which regular expression specifies $\{a, bc\}$?
- and $\{ca, cb\}$?

and an algebra of regular expressions:

$$\begin{aligned}(E_1 + E_2) + E_3 &= E_1 + (E_2 + E_3)\\ (E_1 + E_2)\,E_3 &= E_1\,E_3 + E_2\,E_3\\ E_1\,(E_2\,E_1)^* &= (E_1\,E_2)^*\,E_1\end{aligned}$$

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After thoughts

(from the two examples of reactive systems discussed)

- characterise notions of observation and interaction
- syntax (support for modeling) and semantics (basis for calculation)

After thoughts

... need more general models and theories:

- Several interaction points ($\neq$ functions)
- Non termination (no final states as in automata)
- Need to distinguish normal from anomolous termination (eg deadlock)
- Non determinisim should be taken seriously: the notion of equivalence based on accepted language is blind wrt non determinism

Labelled Transition System

Relational characterization

A LTS over a set A of names is a pair

$$\langle U, \ {}_{\alpha}\!\longleftarrow : A \times U \longleftarrow U \rangle$$

where

- $U = \{u_0, u_1, u_2, ...\}$ is a set of states
- ${}_{\alpha}\!\longleftarrow : A \times U \longleftarrow U$ is an arrow in Rel, capturing the transition relation,

  often given as an A-indexed family of binary relations

$$u' \ {}_{\alpha}\!\overset{a}{\longleftarrow} \ u \ \Leftrightarrow \ \langle a, u' \rangle \ {}_{\alpha}\!\longleftarrow \ u$$

  or simply,

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Transposition

The power-transpose

Binary relations and powerset valued functions are equivalent: each other determines the other uniquely.
The existence and uniqueness of such a transformation leads to the identification of a transpose operator Λ characterized by the following universal property:

$$f = \Lambda R \;\Leftrightarrow\; (yRx \Leftrightarrow y \in f\,x)$$

for relation $R : Y \longleftarrow X$ and function $f : \mathcal{P}Y \longleftarrow X$ or, in a completely pointfree formulation

$$f = \Lambda R \;\Leftrightarrow\; R = \in \cdot f$$

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Transposition

$$f = \Lambda R \;\Leftrightarrow\; R = \in \cdot f$$

Properties

Cancellation	$\in \cdot \Lambda R \;=\; R$
Reflexivity	$\Lambda \in \, = \, \in$
Fusion - a	$\Lambda(f \cdot R) \;=\; \mathcal{P}f \cdot \Lambda R$
Fusion - b	$\Lambda(R \cdot f) \;=\; \Lambda R \cdot f$

Labelled Transition System

Transposing ${}_{\alpha}\!\longleftarrow$

through

$$\alpha = \Lambda \ {}_{\alpha}\!\longleftarrow \quad \Leftrightarrow \quad {}_{\alpha}\!\longleftarrow = \in \cdot \alpha$$

gives rise to a coalgebra

$$\alpha : \mathcal{P}(A \times U) \longleftarrow U$$

in Set for functor $\mathsf{T}X = \mathcal{P}(A \times X)$.

Labelled Transition System

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Labelled Transition System

Transposition also applies to morphisms

A morphism $h : \beta \longleftarrow \alpha$ is a function $h : V \longleftarrow U$ st the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & \mathcal{P}(A \times U) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(\mathrm{id} \times h)} \\ V & \xrightarrow{\beta} & \mathcal{P}(A \times V) \end{array}$$

i.e.,

$$\mathcal{P}(\mathrm{id} \times h) \cdot \alpha \;=\; \beta \cdot h$$

or, going pointwise,

$$\{\langle a, h\,x\rangle \mid \langle a, x\rangle \in \alpha\,u\} \;=\; \beta\,(h\,u)$$

Labelled Transition System

but $\mathcal{P}(\mathsf{id} \times h) \cdot \alpha \;=\; \beta \cdot h$

has the following relational counterpart:

$$(\mathsf{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \;=\; {}_{\beta}\!\longleftarrow \cdot h$$

because

Labelled Transition System

$$
\begin{array}{ll}
 & (\mathrm{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \;=\; {}_{\beta}\!\longleftarrow \cdot h \\
\Leftrightarrow & \qquad \{ \text{ transpose is a isomorphism } \} \\
 & \Lambda((\mathrm{id} \times h) \cdot {}_{\alpha}\!\longleftarrow) \;=\; \Lambda({}_{\beta}\!\longleftarrow \cdot h) \\
\Leftrightarrow & \qquad \{ \; \Lambda(f \cdot R) = \mathcal{P}f \cdot \Lambda R \;\text{ e }\; \Lambda(R \cdot f) = \Lambda R \cdot f \; \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda({}_{\alpha}\!\longleftarrow) \;=\; \Lambda({}_{\beta}\!\longleftarrow) \cdot h \\
\Leftrightarrow & \qquad \{ \text{ definition } {}_{\alpha}\!\longleftarrow \; \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda(\in \cdot\, \alpha) \;=\; \Lambda(\in \cdot\, \beta) \cdot h \\
\Leftrightarrow & \qquad \{ \; \Lambda(R \cdot f) = \Lambda R \cdot f \; \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda(\in) \cdot \alpha \;=\; \Lambda(\in) \cdot \beta \cdot h \\
\Leftrightarrow & \qquad \{ \; \Lambda(\in) = \mathrm{id} \; \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \alpha \;=\; \beta \cdot h
\end{array}
$$

Labelled Transition System

Equality

$$(\mathsf{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \;=\; {}_{\beta}\!\longleftarrow \cdot h$$

can be re-written in terms of an A-indexed family of binary relations:

$$h \cdot {}_{\alpha}\!\xleftarrow{a} \;=\; {}_{\beta}\!\xleftarrow{a} \cdot h$$

which can be decomposed in

$$h \cdot {}_{\alpha}\!\xleftarrow{a} \;\subseteq\; {}_{\beta}\!\xleftarrow{a} \cdot h \tag{1}$$

$${}_{\beta}\!\xleftarrow{a} \cdot h \;\subseteq\; h \cdot {}_{\alpha}\!\xleftarrow{a} \tag{2}$$

Labelled Transition System

Equality

$$(\mathsf{id} \times h) \cdot {}_{\alpha}\!\longleftarrow \;=\; {}_{\beta}\!\longleftarrow \cdot h$$

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which can be decomposed in

$$h \cdot {}_{\alpha}\!\xleftarrow{a} \;\subseteq\; {}_{\beta}\!\xleftarrow{a} \cdot h \qquad (1)$$

$${}_{\beta}\!\xleftarrow{a} \cdot h \;\subseteq\; h \cdot {}_{\alpha}\!\xleftarrow{a} \qquad (2)$$

Going pointwise ...

Transition preservation

$$h \cdot {}_{\alpha}\!\xleftarrow{a} \;\subseteq\; {}_{\beta}\!\xleftarrow{a} \cdot h$$

$\Leftrightarrow$ { *shunting* }

$${}_{\alpha}\!\xleftarrow{a} \;\subseteq\; h^{\circ} \cdot {}_{\beta}\!\xleftarrow{a} \cdot h$$

$\Leftrightarrow$ { introducing variables }

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u' \ {}_{\alpha}\!\xleftarrow{a}\ u \ \Rightarrow\ u' \,(h^{\circ} \cdot {}_{\beta}\!\xleftarrow{a} \cdot h)\, u \rangle$$

$\Leftrightarrow$ { relating-functional-images rule }

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u' \ {}_{\alpha}\!\xleftarrow{a}\ u \ \Rightarrow\ h\, u' \ {}_{\beta}\!\xleftarrow{a}\ h\, u \rangle$$

Going pointwise ...

Transition reflection

$$
\begin{array}{ll}
 & {}_\beta\overset{a}{\longleftarrow} \cdot h \;\subseteq\; h \cdot {}_\alpha\overset{a}{\longleftarrow} \\
\Leftrightarrow & \qquad \{ \text{ introducing variables } \} \\
 & \langle \forall\, u, v' \;:\; u \in U, v' \in V :\; v' \,({}_\beta\overset{a}{\longleftarrow} \cdot h)\, u \;\Rightarrow\; v' \,(h \cdot {}_\alpha\overset{a}{\longleftarrow})\, u \rangle \\
\Leftrightarrow & \qquad \{ \text{ relating-functional-images rule and relational composition } \} \\
 & \langle \forall\, u, v' \;:\; u \in U, v' \in V :\; v' \;{}_\beta\overset{a}{\longleftarrow}\; h\, u \;\Rightarrow \\
 & \qquad \langle \exists\, u' \;:\; u' \in U :\; u' \;{}_\alpha\overset{a}{\longleftarrow}\; u \;\wedge\; v' = h\, u' \rangle \rangle
\end{array}
$$

Simulation

Intuition

A state v simulates another state u (in the same or in a different LTS) if every transition from v is corresponded by a transition from u and this capacity is kept along the whole life of the system to which state space v belongs to.

Simulation

Definition

Given ${}_{\alpha}\!\longleftarrow\, : U \times A \longleftarrow U$ and ${}_{\beta}\!\longleftarrow\, : V \times A \longleftarrow V$ both over A, a **simulation** of ${}_{\alpha}\!\longleftarrow$ in ${}_{\beta}\!\longleftarrow$ is a relation $S : V \longleftarrow U$ such that

$$\forall_{a \in A} \forall_{u \in U, v \in V} \,.\, vSu \;\Rightarrow\; (\forall_{u' \in U} \,.\, u' \;{}_{\alpha}\!\overset{a}{\longleftarrow}\; u \Rightarrow (\exists_{v' \in V} \,.\, v' \;{}_{\beta}\!\overset{a}{\longleftarrow}\; v \wedge v' S u'))$$

Simulation

Definition

Given $_{\alpha}\!\!\longleftarrow\, : U \times A \longleftarrow U$ and $_{\beta}\!\!\longleftarrow\, : V \times A \longleftarrow V$ both over A, a simulation of $_{\alpha}\!\!\longleftarrow$ in $_{\beta}\!\!\longleftarrow$ is a relation $S : V \longleftarrow U$ such that

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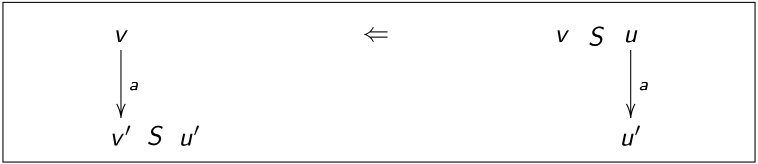
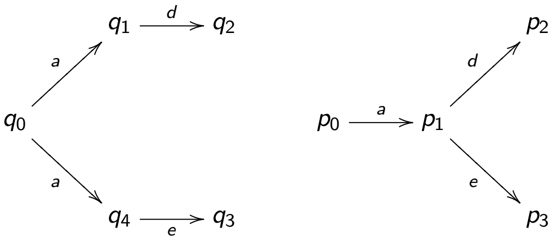


Example



$q_0 \lesssim p_0$ cf. $\{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$

Simulation

Lemma

A relation $S : V \longleftarrow U$ is a simulation of $_{\alpha}\!\longleftarrow$ in $_{\beta}\!\longleftarrow$ iff, for all $a \in A$

$$S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S$$

Properties

because

$$
\begin{array}{ll}
& \forall_{a \in A, u \in U, v \in V} \,.\; vSu \;\Rightarrow \\
& \quad (\forall_{u' \in U} \,.\; u' \;{}_{\alpha}\!\xleftarrow{a}\; u \Rightarrow (\exists_{v' \in V} \,.\; v' \;{}_{\beta}\!\xleftarrow{a}\; v \wedge v' S u')) \\
\Leftrightarrow & \qquad \{ \text{ composition } \} \\
& \forall_{a \in A, u \in U, v \in V} \,.\; vSu \;\Rightarrow\; (\forall_{u' \in U} \,.\; u \xrightarrow{a}_{\alpha} u' \Rightarrow v \,(\xrightarrow{a}_{\beta} \cdot S)\, u' \\
\Leftrightarrow & \qquad \{ \text{ left relational division } \} \\
& \forall_{a \in A, u \in U, v \in V} \,.\; vSu \;\Rightarrow\; v \,((\xrightarrow{a}_{\beta} \cdot S)/ \xrightarrow{a}_{\alpha})\, u \\
\Leftrightarrow & \qquad \{ \text{ going pointfree } \} \\
& S \subseteq (\xrightarrow{a}_{\beta} \cdot S)/ \xrightarrow{a}_{\alpha} \\
\Leftrightarrow & \qquad \{ \text{ Galois connection: } (\cdot R) \dashv (/R) \} \\
& S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S
\end{array}
$$

Properties

Lemma

1. The identity relation id and the empty relation is a simulation
2. The composition $S \cdot R$ of two simulations is a simulation
3. The union $S \cup R$ of two simulations is a simulation

Properties

because

1.

$$\bot \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\beta \cdot \bot \;\wedge\; \mathrm{id} \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\alpha \cdot \mathrm{id}$$

$\Leftrightarrow$ { $\bot$ and id are absorbing and identity for composition }

true

2.

$$(S \cdot R) \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\beta \cdot (S \cdot R)$$

$\Leftarrow$ { $S \cdot \xrightarrow{a}_\gamma \subseteq \xrightarrow{a}_\beta \cdot S$, $\cdot$-assoc, monotony }

$$(S \cdot R) \cdot \xrightarrow{a}_\alpha \subseteq S \cdot \xrightarrow{a}_\gamma \cdot R$$

$\Leftarrow$ { $R \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\gamma \cdot R$, $\cdot$-assoc, monotony }

$$(S \cdot R) \cdot \xrightarrow{a}_\alpha \subseteq (S \cdot R) \cdot \xrightarrow{a}_\alpha$$

Properties

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$$(S \cdot R) \cdot \xrightarrow{a}_{\alpha} \subseteq (S \cdot R) \cdot \xrightarrow{a}_{\alpha}$$

Properties

3.

$$(S \cup R) \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\beta \cdot (S \cup R)$$

$\Leftrightarrow$ { $(R\cdot)$ and $(\cdot R)$ preserve $\cup$ as lower adjoints }

$$(S \cdot \xrightarrow{a}_\alpha \cup R \cdot \xrightarrow{a}_\alpha) \subseteq (\xrightarrow{a}_\beta \cdot S \cup \xrightarrow{a}_\beta \cdot R)$$

$\Leftarrow$ { $\cup$ definition }

$$S \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\beta \cdot S \;\wedge\; R \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\beta \cdot R$$

$\Leftrightarrow$ { hipotheses }

true

Similarity

Definition

$$p \lesssim q \;\Leftrightarrow\; \langle \exists\, R \;::\; R \text{ is a simulation and } \langle p, q\rangle \in R \rangle$$

Lemma

The similarity relation is a preorder
(ie, reflexive and transitive)

because

By definition $\lesssim$ is the greatest simulation. Then (why?), $\lesssim \cdot \lesssim \subseteq \lesssim$ and id $\subseteq \lesssim$.

Bisimulation

Definition

A relation $S : V \longleftarrow U$ over the state spaces of ${}_{\alpha}\!\longleftarrow : U \times A \longleftarrow U$ and ${}_{\beta}\!\longleftarrow : V \times A \longleftarrow V$ is a bisimulation iff both S and S° are simulations

i.e.

$$S \cdot \xrightarrow{a}_{\alpha} \subseteq \xrightarrow{a}_{\beta} \cdot S \quad \wedge \quad {}_{\beta}\!\xleftarrow{a} \cdot S \subseteq S \cdot {}_{\alpha}\!\xleftarrow{a}$$

for all $a \in A$.

Bisimulation

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for all $a \in A$.

Bisimulation

because

The first conjunct defines S as a simulation.
The second one is derived as follows:

$$
\begin{array}{ll}
 & S^\circ \quad \text{is a simulation} \\
\Leftrightarrow & \quad \{ \ \text{definition of simulation} \ \} \\
 & S^\circ \cdot \xrightarrow{a}_\beta \ \subseteq \ \xrightarrow{a}_\alpha \cdot S^\circ \\
\Leftrightarrow & \quad \{ \ (\xrightarrow{a}_\gamma)^\circ = \ {}_\gamma\xleftarrow{a} \ \} \\
 & S^\circ \cdot ({}_\beta\xleftarrow{a})^\circ \ \subseteq \ ({}_\alpha\xleftarrow{a})^\circ \cdot S^\circ \\
\Leftrightarrow & \quad \{ \ (R \cdot S)^\circ = S^\circ \cdot R^\circ \ \} \\
 & ({}_\beta\xleftarrow{a} \cdot S)^\circ \ \subseteq \ (S \cdot {}_\alpha\xleftarrow{a})^\circ \\
\Leftrightarrow & \quad \{ \ \text{monotonicity: } R \subseteq S \Leftrightarrow R^\circ \subseteq S^\circ \ \} \\
 & {}_\beta\xleftarrow{a} \cdot S \ \subseteq \ S \cdot {}_\alpha\xleftarrow{a}
\end{array}
$$

Bisimulation

going pointwise

$$
\begin{array}{ll}
 & {}_{\beta}\xleftarrow{a} \cdot S \subseteq S \cdot {}_{\alpha}\xleftarrow{a} \\
\Leftrightarrow & \quad \{ \text{ Galois: } (R\cdot) \dashv (R\backslash) \ \} \\
 & S \subseteq {}_{\beta}\xleftarrow{a} \backslash (S \cdot {}_{\alpha}\xleftarrow{a}) \\
\Leftrightarrow & \quad \{ \text{ introducing variables } \} \\
 & \forall_{v \in V, u \in U} \,.\; vSu \Rightarrow v\,({}_{\beta}\xleftarrow{a} \backslash (S \cdot {}_{\alpha}\xleftarrow{a}))\, u \\
\Leftrightarrow & \quad \{ \text{ definition of left division } \backslash \ \} \\
 & \forall_{v \in V, u \in U} \,.\; vSu \Rightarrow (\forall_{v' \in V} \,.\; v' \ {}_{\alpha}\xleftarrow{a}\ v \Rightarrow v'\,({}_{\beta}\xleftarrow{a} \cdot S)\, u') \\
\Leftrightarrow & \quad \{ \text{ definition of } \cdot \ \} \\
 & \forall_{v \in V, u \in U} \,.\; vSu \Rightarrow (\forall_{v' \in V} \,.\; v' \ {}_{\beta}\xleftarrow{a}\ v \Rightarrow (\exists_{u' \in U} \,.\; u' \ {}_{\alpha}\xleftarrow{a}\ u \wedge v' S u'))
\end{array}
$$

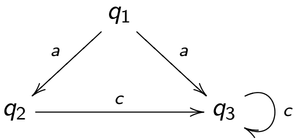
Bisimulation

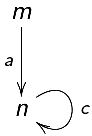
The Game characterization

Two players R and I discuss whether the transition structures are mutually corresponding

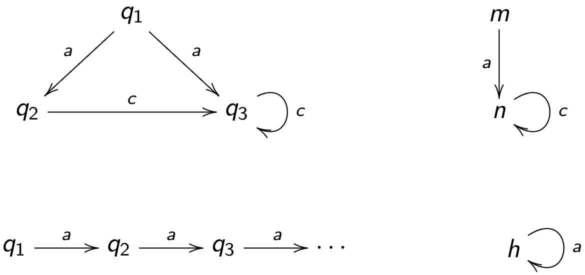
- R starts by chosing a transition
- I replies trying to match it
- if I succeeds, R plays again
- R wins if I fails to find a corresponding match
- I wins if it replies to all moves from R and the game is in a configuration where all states have been visited or R can't move further. In this case is said that I has a wining strategy

Examples





Examples



Properties

Lemma

The graph of a coalgebra morphism $h : \beta \longleftarrow \alpha$, i.e., h itself regarded as a binary relation, is a bisimulation.

Properties

because (partially ...)

$$h \cdot {}_{\alpha}\overset{a}{\longleftarrow} \subseteq {}_{\beta}\overset{a}{\longleftarrow} \cdot h$$

$\Leftrightarrow$ { shunting }

$${}_{\alpha}\overset{a}{\longleftarrow} \subseteq h^{\circ} \cdot {}_{\beta}\overset{a}{\longleftarrow} \cdot h$$

$\Leftrightarrow$ { monotonicity }

$$({}_{\alpha}\overset{a}{\longleftarrow})^{\circ} \subseteq (h^{\circ} \cdot {}_{\beta}\overset{a}{\longleftarrow} \cdot h)^{\circ}$$

$\Leftrightarrow$ { converse }

$$\overset{a}{\longrightarrow}_{\alpha} \subseteq h^{\circ} \cdot \overset{a}{\longrightarrow}_{\beta} \cdot h$$

$\Leftrightarrow$ { function-relation law }

$$h \cdot \overset{a}{\longrightarrow}_{\alpha} \subseteq \overset{a}{\longrightarrow}_{\beta} \cdot h$$

Properties

Lemma

The converse of a bisimulation $S : V \longleftarrow U$ is still a bissimulation.

Properties

because

S° is bisimulation

$\Leftrightarrow$ { definition of bisimulation }

$$S^\circ \cdot \xrightarrow{a}_\alpha \subseteq \xrightarrow{a}_\beta \cdot S^\circ \;\wedge\; {}_\beta\xleftarrow{a} \cdot S^\circ \subseteq S^\circ \cdot {}_\alpha\xleftarrow{a}$$

$\Leftrightarrow$ { $(\xrightarrow{a}_\gamma)^\circ = {}_\gamma\xleftarrow{a}$ }

$$S^\circ \cdot ({}_\alpha\xleftarrow{a})^\circ \subseteq ({}_\beta\xleftarrow{a})^\circ \cdot S^\circ \;\wedge\; (\xrightarrow{a}_\beta)^\circ \cdot S^\circ \subseteq S^\circ \cdot (\xrightarrow{a}_\alpha)^\circ$$

$\Leftrightarrow$ { converse of composition }

$$({}_\alpha\xleftarrow{a} \cdot S)^\circ \subseteq (S \cdot {}_\beta\xleftarrow{a})^\circ \;\wedge\; (S \cdot \xrightarrow{a}_\beta)^\circ \subseteq (\xrightarrow{a}_\alpha \cdot S)^\circ$$

$\Leftrightarrow$ { monotonicity }

$${}_\alpha\xleftarrow{a} \cdot S \subseteq S \cdot {}_\beta\xleftarrow{a} \;\wedge\; S \cdot \xrightarrow{a}_\beta \subseteq \xrightarrow{a}_\alpha \cdot S$$

$\Leftrightarrow$ { hipothesis }

true

Bisimilarity

Definition

$$p \sim q \;\Leftrightarrow\; \langle \exists\, R \;::\; R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

Lemma

1. The identity relation id is a bisimulation
2. The empty relation $\perp$ is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \mid i \in I\}$ is a bisimulation

Bisimilarity

Lemma

The bisimilarity relation is an equivalence relation
(ie, reflexive, symmetric and transitive)

Lemma

The class of all bisimulations between two LTS has the structure of a complete lattice, ordered by set inclusion, whose top is the bisimilarity relation $\sim$.

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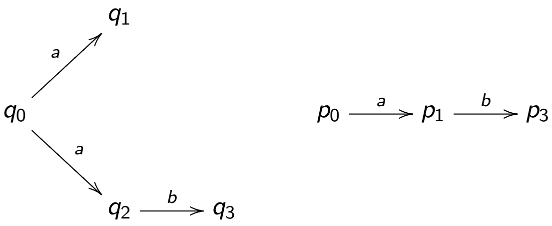
Bisimilarity

Warning

The bisimilarity relation $\sim$ is not the symmetric closure of $\lesssim$

Example

$$q_0 \lesssim p_0,\; p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$



After thoughts

Similarity as the greatest simulation

$$\lesssim \triangleq \bigcup\{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

$$\sim \triangleq \bigcup\{S \mid S \text{ is a bisimulation}\}$$

cf relational translation of definitions
$\lesssim$ and $\sim$ as greatest fix points (Tarski's theorem)

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The questions to follow ...

- We already have a semantic model for reactive systems. With which language shall we describe them?
- How to compare and transform such systems?
- How to express and prove their proprieties?

$\leadsto$ process languages and calculi
cf. CCS (Milner, 80), CSP (Hoare, 85),
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π-*calculus* (Milner, 89), among many others

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