Algebraic and Coalgebraic methods in software development

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Signature morphism

Definition

Let $\Sigma = (S, \Omega)$ and $\Sigma' = (S', \Omega')$ be signatures. A *signature morphism* $\sigma : \Sigma \to \Sigma'$*, is a pair* $\sigma = (\sigma_{sort}, \sigma_{op})$*, where*

- $\sigma_{sorts} : S \to S'$

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Renaming, Adding, Identifying

Definition (Reduct Algebra)

Let $\mathbf{A}'$ *be a* Σ'*-algebra, and* $\sigma : \Sigma \to \Sigma'$ *be a signature morphism. The* *σ-reduct* *of* $\mathbf{A}'$ *is the* Σ*-algebra* $\mathbf{A}' \restriction_\sigma$ *defined as follows:*

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Given a morphism $h' : \mathbf{A}' \to \mathbf{B}'$, the *$\sigma$-reduct de* h' is $h' \restriction_\sigma : A' \restriction_\sigma \to B' \restriction_\sigma$ defined by $(h' \restriction_\sigma)_s = h'_{\sigma(s)}$

Satisfaction lemma

Let $\sigma : \Sigma \to \Sigma'$ be a signature morphism and X a set of variables for Σ.
Take $X'_v = \biguplus \{X_s : \sigma_{sorts}(s) = v\}$

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Extension to terms
$\widehat{\sigma} : \mathrm{T}(\Sigma, \mathrm{X}) \to (T(\Sigma', X'))\restriction_\sigma$

(i) If $t = x : s$, then $\widehat{\sigma}(t) = x : \sigma(s)$;

(ii) If $t = c$, then $\widehat{\sigma}(t) = \sigma(c)$;

(iii) If $t = f(t_1, \ldots, t_n)$, with $f : s_1, \ldots, s_n \to s \in \Sigma$, then
$\widehat{\sigma}(t) = \sigma(f)(\widehat{\sigma}(t_0), \ldots, \widehat{\sigma}(t_n))$.

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Let Σ, Σ' be signatures, $\mathbf{A}'$ be a Σ'-algebra and ϕ be a Σ-equation. Then,

$$\mathbf{A}' \models \sigma(\phi) \text{ iff } \mathbf{A}' \restriction_\sigma \models \phi.$$

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Corollary

$$\Phi \models_\Sigma t_1 \approx t_2 \quad \Rightarrow \quad \sigma(\Phi) \models_{\Sigma'} \sigma(t_1 \approx t_2).$$

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- When the implication "$\Leftarrow$" also holds, the morphism is called conservative.

Translations

Syntax

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Semantics

$$\begin{tikzcd} \Sigma - algebras & \Sigma - homomorphisms \\ \Sigma' - algebras \arrow[u, "\restriction_\sigma"] & \Sigma' - homomorphisms \arrow[u, "\restriction_\sigma"] \end{tikzcd}$$

Structured specifications

We follow Sannella and Tarlecki [ST88], by assuming that the software systems, described by (algebraic) specifications, are adequately represented as models of an appropriated underlying logic. Therefore, a specification describes a signature and a class the models over this signature - *the models of the specification*.

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Definition

A specification SP is a pair $\langle \Sigma, K \rangle$, *where* Σ *is a signature and* K *is a class of* Σ*-algebra. We will represent* Σ *by* $Sig(SP)$ *and* K *by* $\mathrm{Mod}(SP)$ *- the class of models of SP.*

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We build more complex specification from simpler ones following the modular development of programmes.

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3. **translate** - to define a specification over a signature Σ' from a specification over another specification over a signature Σ using a signature morphism $\sigma : \Sigma \to \Sigma'$.
4. **derive (or Hidding)** - to define a specification over a signature Σ from a specification over another specification over a signature Σ' using a signature morphism $\sigma : \Sigma \to \Sigma'$, by considering the reducts.

Basic operators

- **flat**
  - Syntax:

    $< .,. >$: $\mathit{Sig}, \mathit{Sentences} \to \mathit{Spec}$
  - Semantics: Σ a signature and Φ a set of sentences over Σ.

    $\mathit{Sig}(<\Sigma, \Phi>) = \Sigma$

    $\mathrm{Mod}(<\Sigma, \Phi>) =_{\mathit{def}} \{\mathbf{A} \in \mathit{Alg}(\Sigma) | \mathbf{A} \models \Phi\}$

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- **union**

Let SP_1 e SP_2 be specifications over a same signature Σ:

  - Syntax:

    $< . \cup . >$: *Spec*, *Spec* $\to$ *Spec*
  - Sematics:

    $Sig(SP1 \cup SP2) = Sig(SP_1) = Sig(SP_2)$

    $\mathrm{Mod}(SP1 \cup SP2) =_{def} \mathrm{Mod}(SP_1) \cap \mathrm{Mod}(SP_2)$

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- **translate**
  - Syntax:

    translate . **by** . : *Spec*, *morph* $\to$ *Spec*
  - Semantics: let $\sigma : \Sigma \to \Sigma'$ be a signature morphism and *SP* a specification with $Sig(SP) = \Sigma$.

    $Sig(\textbf{translate } SP \textbf{ by } \sigma) =_{def} \Sigma'$

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A **structured specification** is a specification *SP* obtained by a finite number of applications o these 4 operators.

Equational case

Not all algebraic specification (classe of algebras) can be axiomatized by a set of equations.

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A specification is flat iff the class of algebras is closed by subalgebras, homomorphic images and products.

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Even with first-order formulas it is impossible!

More useful operators

- **enrich**: To add new sorts, new axioms and new operation symbols
Let $\Sigma = (S, \Omega)$, $\Sigma' = (S \cup S', \Omega \cup \Omega')$ and $\iota : \Sigma \hookrightarrow \Sigma'$ the inclusion morphism.

$$\textbf{enrich } SP \textbf{ by sorts } S' \textbf{ opns } F' \textbf{ axioms } \Phi' = (\textbf{translate } SP \textbf{ by } \iota) \cup \langle \Sigma', \Phi' \rangle$$

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- **export**: A particular case of **derive**, the morphism is the inclusion, i.e., let $\iota : \Sigma \hookrightarrow \Sigma'$:

$$\textbf{export } \Sigma' \textbf{ from } SP = \textbf{ derive from } SP \textbf{ by } \iota.$$

Reach operator

- A *reachability constraint of* Σ is a pair $\mathcal{R} = \langle S_{\mathcal{R}}, F_{\mathcal{R}} \rangle$ s.t. $F_{\mathcal{R}} \subseteq \Omega$ and $S_{\mathcal{R}} = \{s \in S |$ existe um $f \in (F_{\mathcal{R}})_{ws}\}$. ▶ An $s \in S_{\mathcal{R}}$ is called a *constrained sort* and a symbol $f \in F_{\mathcal{R}}$ *a constructor*.

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- A *constructor term* is a $t \in T(\Sigma', X')_s$, where $\Sigma' = \langle S, F_{\mathcal{R}} \rangle$, $X' = X_s$ if $s \in S \setminus S_{\mathcal{R}}$, and $X'_s = \emptyset$ if $s \in S_{\mathcal{R}}$.

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A Σ-algebra $\mathbf{A}$, satisfies a *reachability constraint* $\mathcal{R} = \langle S_\mathcal{R}, F_\mathcal{R}\rangle$, $\mathbf{A} \models \mathcal{R}$, if for all $s \in S$ and every $a \in A_s$, there exists a constructor term t and an evaluation $\alpha : X' \to A$ s.t. $\alpha(t) = a$.

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Theorem

Let $\mathbf{A}$ *be a* Σ*-algebra and* $\mathcal{R}$ *a reachability constraint over* Σ*. TFAE*

1. $\mathbf{A} \models \mathcal{R}$
2. *for every* $s \in S$*, and any* $a \in A_s$ *there exists a constructor term* t *of sort* S *such that* $\mathbf{A}, \alpha \models \exists Var(t).x = t$*, where* $x \in X_s$*,* $x \notin Var(t)$ *and* $\alpha : X \to A$ *an evaluation such that* $\alpha(x) = a$*.*

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3. *For all* $s \in S$*,*
$$\mathbf{A} \models (\forall x : s) \bigvee_{t \in (T_{\mathcal{R}})_s} \exists Var(t)\, x \approx t.$$

Reach operator

reach

- Syntax: **reach** **with** . : *Spec*, *Opns* $\rightarrow$ *Spec*
- Semantics:

  Let $\mathcal{R} = \langle S_{\mathcal{R}}, F_{\mathcal{R}} \rangle$ a *reachability constraint* over $Sig(SP)$

  $Sig(\textbf{reach } SP \textbf{ with } F_{\mathcal{R}}) = Sig(SP)$

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Example

$INTZERO =$ **reach** INT **with**
$F_{\mathcal{R}} =$
$0 : \rightarrow\ int;$
$s,p:\ int \rightarrow int;$

Examples [ST88]

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\begin{array}{lll}
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 & \textbf{opns} & \mathit{true} : \mathit{bool} \\
 & & \mathit{false} : \mathit{bool} \\
 & \textbf{axioms} & \mathit{true} \neq \mathit{false} \\
 & & \forall x{:}\mathit{bool}.\, x = \mathit{true} \vee x = \mathit{false}
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\begin{array}{lll}
INT = & \multicolumn{2}{l}{\textbf{enrich } BOOL \textbf{ by}} \\
 & \textbf{sorts} & int \\
 & \textbf{opns} & 0 : int \\
 & & succ : int \rightarrow int \\
 & & pred : int \rightarrow int \\
 & \textbf{axioms} & \ldots \text{induction scheme for } int \ldots \\
 & & \forall x{:}int.\, pred(x) \neq x \wedge succ(x) \neq x \\
 & & \forall x{:}int.\, pred(succ(x)) = x \wedge succ(pred(x)) = x
\end{array}
$$

Examples [ST88]

$$
\begin{array}{lll}
BOOL = & \textbf{sorts} & bool \\
& \textbf{opns} & true : bool \\
& & false : bool \\
& \textbf{axioms} & true \neq false \\
& & \forall x{:}bool.\, x = true \vee x = false
\end{array}
$$

$$
\begin{array}{lll}
INT = & \multicolumn{2}{l}{\textbf{enrich } BOOL \textbf{ by}} \\
& \textbf{sorts} & int \\
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\end{array}
$$

$$
\begin{array}{lll}
INTORD = & \multicolumn{2}{l}{\textbf{enrich } INT \textbf{ by}} \\
& \textbf{opns} & po : int \times int \to bool \\
& \textbf{axioms} & \forall x{:}int.\, po(x,x) = true \\
& & \forall x, y{:}int.\, po(x,y) = true \wedge po(y,x) = true \Longrightarrow x = y \\
& & \forall x, y, z{:}int.\, po(x,y) = true \wedge po(y,z) = true \Longrightarrow po(x,z) = true
\end{array}
$$

EXAMPLE [ST88]

$$
\begin{array}{lll}
\mathit{INTLIST} = & \multicolumn{2}{l}{\textbf{enrich } \mathit{INTORD} \textbf{ by}} \\
& \textbf{sorts} & \mathit{list} \\
& \textbf{opns} & \mathit{nil} : \mathit{list} \\
& & \mathit{cons} : \mathit{int} \times \mathit{list} \to \mathit{list} \\
& & \mathit{head} : \mathit{list} \to \mathit{int} \\
& & \mathit{tail} : \mathit{list} \to \mathit{list} \\
& & \mathit{append} : \mathit{list} \times \mathit{list} \to \mathit{list} \\
& & \mathit{is\_in} : \mathit{int} \times \mathit{list} \to \mathit{bool} \\
& \textbf{axioms} & \ldots \text{induction scheme for } \mathit{list} \ldots \\
& & \forall x{:}\mathit{int}.\, \forall l{:}\mathit{list}.\ \mathit{cons}(x,l) \neq l \\
& & \forall x{:}\mathit{int}.\, \forall l{:}\mathit{list}.\ \mathit{head}(\mathit{cons}(x,l)) = x \\
& & \forall x{:}\mathit{int}.\, \forall l{:}\mathit{list}.\ \mathit{tail}(\mathit{cons}(x,l)) = l \\
& & \forall l{:}\mathit{list}.\ \mathit{append}(\mathit{nil},l) = l \\
& & \forall x{:}\mathit{int}.\, \forall l,l'{:}\mathit{list}.\ \mathit{append}(\mathit{cons}(x,l),l') = \mathit{cons}(x,\mathit{append}(l,l')) \\
& & \forall x{:}\mathit{int}.\ \mathit{is\_in}(x,\mathit{nil}) = \mathit{false} \\
& & \forall x,y{:}\mathit{int}.\, \forall l{:}\mathit{list}.\ \mathit{is\_in}(x,\mathit{cons}(y,l)) = \mathit{true} \iff \\
& & \qquad\qquad (x = y \vee \mathit{is\_in}(x,l) = \mathit{true})
\end{array}
$$

Calculus for Structured specifications

$$\frac{\varphi \in \Phi}{\langle \Sigma, \Phi \rangle \vdash \varphi}$$

$$\frac{SP_1 \vdash \varphi}{SP_1 \cup SP_2 \vdash \varphi}$$

$$\frac{SP_2 \vdash \varphi}{SP_1 \cup SP_2 \vdash \varphi}$$

$$\frac{SP \vdash \varphi}{\textbf{translate } SP \textbf{ by } \sigma \vdash \sigma(\varphi)}$$

$$\frac{SP' \vdash \sigma(\varphi)}{\textbf{derive from } SP' \textbf{ by } \sigma \vdash \varphi}$$

$$\frac{SP \vdash \varphi_1 \ldots SP \vdash \varphi_n \text{ e } \{\varphi_1, \ldots, \varphi_n\} \vdash \varphi}{SP \vdash \varphi}$$

Completeness

$$SP \models \varphi \quad \textit{iff} \quad SP \vdash \varphi.$$

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A more abstract treatment using INSTITUTIONS.

Stepwise refinement process

The *stepwise refinement process* is the systematic process by which, from a specification SP_0 we successively build more restrictive specifications by introducing new requirements:

$$SP_0 \rightsquigarrow SP_1 \rightsquigarrow SP_2 \rightsquigarrow \cdots \rightsquigarrow SP_{n-1} \rightsquigarrow SP_n,$$

where for all $1 \leq i \leq n$, $SP_{i-1} \rightsquigarrow SP_i$ is a refinement.

The software development - the stepwise refinement methodology

Definition (Refinement)

Let SP and SP' be specifications. SP' is a refinement of SP if:

- $Sig(SP) = Sig(SP')$;
- $\mathrm{Mod}(SP') \subseteq \mathrm{Mod}(SP)$;

We write $SP \rightsquigarrow SP'$ when SP' is a refinement of SP.

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Definition (σ-refinement)

Let SP and SP' be algebraic specifications and $\sigma : \mathrm{Sig}(SP) \to \mathrm{Sig}(SP')$. SP' is a σ-refinement of SP, in symbols $SP \rightsquigarrow_\sigma SP'$, if:

- $\mathrm{Mod}(SP') \restriction_\sigma \subseteq \mathrm{Mod}(SP)$,

where $\mathrm{Mod}(SP') \restriction_\sigma = \{\mathbf{A}' \restriction_\sigma | \mathbf{A}' \in \mathrm{Mod}(SP')\}$.

Compositionality

Vertical composition

$SP \rightsquigarrow_{\sigma} SP' \rightsquigarrow_{\phi} SP''$
$\mathrm{Mod}(SP'') \restriction_{\phi \circ \sigma} \subseteq \mathrm{Mod}(SP') \restriction_{\sigma} \subseteq \mathrm{Mod}(SP)$

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Stepwise Refinement Process:

$$SP_0 \rightsquigarrow_{\sigma_0} SP_1 \rightsquigarrow_{\sigma_1} SP_2 \rightsquigarrow_{\sigma_2} \ldots \rightsquigarrow_{\sigma_{n-2}} SP_{n-1} \rightsquigarrow_{\sigma_{n-1}} SP_n.$$

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Horizontal composition

$$\frac{SP_1 \rightsquigarrow SP'_1, \ldots, SP_n \rightsquigarrow SP'_n}{op(SP_1, \ldots, SP_n) \rightsquigarrow op(SP'_1, \ldots, SP'_n)}$$

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Horizontal composition - not so easy!

Horizontal composition

Theorem

Let $\Sigma \subseteq \Sigma'$ *and suppose* $SP_0 \rightsquigarrow_\iota SP'_0$ *and* $SP_1 \rightsquigarrow_\iota SP'_1$*, and* $\phi : \Sigma' \to \Sigma''$ *a signature morphisms. Then*

1. $SP_0 \cup SP_1 \rightsquigarrow_\iota SP'_0 \cup SP'_1$;
2. $\textbf{translate } SP_0 \textbf{ by } \phi\restriction_\Sigma \rightsquigarrow_\iota \textbf{translate } SP'_0 \textbf{ by } \phi$;

Limitations of the classical approach

spec SPEC1 =
sorts
s;
ops
$f : s \to s$;
Ax + Ir

$$t \approx t; \quad \frac{t \approx t'}{t' \approx t} \quad \frac{t \approx t', t' \approx t''}{t \approx t''}; \quad \frac{t \approx t'}{f(t) \approx f(t')}$$

spec SPEC2 =
sorts
s;
ops
$ok :\to s, f : s \to s, test : s \times s \to s$;
Ax + Ir

$$test(t,t) \approx ok; \quad \frac{test(t,t') \approx ok}{test(t',t) \approx ok}; \quad \frac{test(t,t') \approx ok, test(t',t'') \approx ok}{test(t,t'') \approx ok}; \quad \frac{test(t,t') \approx ok}{test(f(t),f(t')) \approx ok};$$

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$\dfrac{test(t, t') \approx ok, test(t', t'') \approx ok}{test(t, t'') \approx ok}$;
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- Naturally, $\mathrm{SPEC1} \models \varphi \approx \varphi'$ iff $\mathrm{SPEC2} \models test(\varphi, \varphi') \approx ok$
- However, $\iota : Sig(\mathrm{SPEC1}) \to Sig(\mathrm{SPEC2})$ is the unique morphism definable between the specifications of SPEC1 and SPEC2.

Motivations

Refinement based on signature morphisms

- a formula is mapped into another one;
- formula structure is preserved;

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The strategy

- Introduce a formalization of the refinement where the translation of specifications is witnessed by a suitable kind of multifunctions;

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The strategy

- Introduce a formalization of the refinement where the translation of specifications is witnessed by a suitable kind of multifunctions;
- Generalize this approach by allowing translations between specifications expressed in logics with different dimensions;

Interpretations within algebraic specification

Refinement by interpretations

A translation $\tau : \mathrm{Eq}(\Sigma) \to \mathcal{P}(\mathrm{Eq}(\Sigma'))$ *interprets* SP if there is a specification SP' over Σ' such that:

- for all $t \approx t' \in \mathrm{Eq}(\mathrm{Sig}(SP)), SP \models t \approx t'$ iff $SP' \models \tau(t \approx t')$

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A mathematical example

The self translation $\tau(t \approx t') = \{\neg\neg t \approx \neg\neg t'\}$ interprets the specification $\mathbb{BA}$ (boolean algebras) in the specification $\mathbb{HA}$ (Heyting algebras).

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Definition

SP' is a refinement by the interpretation τ of SP if

- *τ interprets SP and*
- *for all $t \approx t' \in \mathrm{Eq}(\mathrm{Sig}(SP)), SP \models t \approx t'$ implies $SP' \models \tau(t \approx t')$*

Ex. BAMS: replacing operations by atomic transactions

Σ_1 :

sorts

$Ac; Int;$

ops

$bal : Ac \to Int;$

$cred, deb : Ac \times Int \to Ac$

spec BAMS = enrich EQ_{Σ_1}
and INT with

axioms

$bal(cred(x, n)) \approx bal(x) + n;$

$bal(deb(x, n)) \approx bal(x) + (-n).$

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Σ_2 :

sorts

$Ac; Int;$

ops

...

$val : Ac \to Ac$

spec $\mathrm{BAMS2}$ = enrich EQ_{Σ_2} and INT with

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$\tau : \mathrm{Eq}(\Sigma_1) \to \mathcal{P}(\mathrm{Eq}(\Sigma_2)) = \{\langle op(x), y\rangle \to \{\langle val(op(x)), y\rangle\} | op \in \{cred, deb\}\}$

Ex. NatBool: encapsulating sorts

- **Spec Nat=** enrich $EQ_{\Sigma_{Nat}}$ by

ops $s : nat \to nat$;

IR $\frac{s(x) \approx s(y)}{x \approx y}$

- **Spec NatEq=** enrich BOOL by

sorts nat;

ops $s : nat \to nat; eq : nat, nat \to bool$;

axioms $eq(x, x) \approx true$

IR $\frac{eq(x, y) \approx true}{eq(y, x) \approx true}$; $\frac{eq(x, y) \approx true\ ,\ eq(y, z) \approx true}{eq(x, z) \approx true}$;

$\frac{eq(x, y) \approx true}{eq(s(x), s(y)) \approx true}$; $\frac{eq(s(x), s(y)) \approx true}{eq(x, y) \approx true}$.

Ex. NatBool: encapsulating sorts

- Spec Nat= enrich $EQ_{\Sigma_{Nat}}$ by

ops $s : nat \to nat;$

IR $\dfrac{s(x) \approx s(y)}{x \approx y}$

- Spec NatEq= enrich BOOL by

sorts $nat;$

ops $s : nat \to nat; eq : nat, nat \to bool;$

axioms $eq(x,x) \approx true$

IR $\dfrac{eq(x,y) \approx true}{eq(y,x) \approx true}; \quad \dfrac{eq(x,y) \approx true\ ,\ eq(y,z) \approx true}{eq(x,z) \approx true};$

$\dfrac{eq(x,y) \approx true}{eq(s(x),s(y)) \approx true}; \quad \dfrac{eq(s(x),s(y)) \approx true}{eq(x,y) \approx true}.$

Taking $\tau(x : nat \approx y : nat) = \{eq(x : nat, y : nat) \approx true\}$, we have

$$\text{Nat} \to_\tau \text{NatEq}$$

k-logics

Goal

Provide a suitable context to deal simultaneously with different specification logics as, assertional, equational, modal, ...

- Let Σ be a signature and Va a set of variables for Σ. The set of terms in the variables Va over Σ is denoted by $\mathrm{Fm}_\Sigma(\mathrm{Va})$.

Definition

A k-logic is a pair $\mathcal{L} = \langle \Sigma, \vdash_\mathcal{L} \rangle$, where Σ is a signature and $\vdash_\mathcal{L} \subseteq \mathcal{P}(\mathrm{Fm}^k_\Sigma(\mathrm{Va})) \times \mathrm{Fm}^k_\Sigma(\mathrm{Va})$ a relation such for all $\Gamma \cup \Delta \cup \{\bar{\gamma}, \bar{\varphi}\} \subseteq \mathrm{Fm}^k_\Sigma(\mathrm{Va})$:

(i) $\Gamma \vdash_\mathcal{L} \bar{\gamma}$ *for each* $\bar{\gamma} \in \Gamma$;

(ii) *if* $\Gamma \vdash_\mathcal{L} \bar{\varphi}$, *and* $\Delta \vdash_\mathcal{L} \bar{\gamma}$ *for each* $\bar{\gamma} \in \Gamma$, *then* $\Delta \vdash_\mathcal{L} \bar{\varphi}$;

(iii) *if* $\Gamma \vdash_\mathcal{L} \bar{\varphi}$, *then* $\sigma(\Gamma) \vdash_\mathcal{L} \sigma(\bar{\varphi})$ *for every substitution* σ.

Semantics

A pair $\mathcal{A} = \langle \mathbf{A}, F \rangle$ is a *k-data structure over* Σ if

- $\mathbf{A}$ is a Σ-algebra over Σ
- F is a subset of A^k.

Semantic consequence

$\Gamma \models_{\mathcal{A}} \bar{\varphi}$ if for any assignment $h : \mathrm{Va} \to A$, $h(\Gamma) \subseteq F$ implies $h(\bar{\varphi}) \in F$.

Familiar examples

1-data structures: models of CPC, e.g. $\mathcal{A} = \langle \mathbf{A}, F \rangle$ over a sentential language with A a Boolean algebra and $F = \{\top\}$;

2-data structures: models of the (free) equational logic over Σ, e.g. $\mathcal{A} = \langle \mathbf{A}, F \rangle$ over a multi-sorted signature with $F = \ id_A$;

Translating k-logics

Definition ((k, m)-translation from Σ to Σ')

$$\tau : \mathrm{Fm}^{k}_{\Sigma}(\mathrm{Va}) \to \mathcal{P}(\mathrm{Fm}^{m}_{\Sigma'}(\mathrm{Va}))$$

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$$\tau : \mathrm{Fm}^k_\Sigma(\mathrm{Va}) \to \mathcal{P}(\mathrm{Fm}^m_{\Sigma'}(\mathrm{Va}))$$

Definition (Interpretation)

τ interprets $\mathcal{L}$ if there is a m-logic $\mathcal{L}'$ over Σ' such that, for any $\Gamma \cup \{\bar{\varphi}\} \subseteq \mathrm{Fm}^k_\Sigma(\mathrm{Va})$,

$$\Gamma \vdash_{\mathcal{L}} \bar{\varphi} \textit{ iff } \tau(\Gamma) \vdash_{\mathcal{L}'} \tau(\bar{\varphi}).$$

A paradigmatic example

$$\rho(\langle p, q\rangle) = \{\langle p \to q\rangle, \langle q \to p\rangle\}$$

$$\begin{tikzcd}
\color{blue}{\mathrm{BA}} \arrow[r, bend left] & \color{blue}{\mathrm{CPC}} \arrow[l, bend left]
\end{tikzcd}$$

$$\tau(\langle p\rangle) = \{\langle p, \top\rangle\}$$

τ-model class

Definition (τ-model)

Let $\tau : \mathrm{Fm}^k_\Sigma(\mathrm{Va}) \to \mathcal{P}(\mathrm{Fm}^m_{\Sigma'}(\mathrm{Va}))$ *and* $\mathcal{L}$ *over* Σ*. An I-data structure* $\mathcal{A}$ *is a* *τ-model of $\mathcal{L}$* *if for any* $\Gamma \cup \{\bar{\varphi}\} \subseteq \mathrm{Fm}^k_\Sigma(\mathrm{Va})$,

$$\Gamma \vdash_{\mathcal{L}} \bar{\varphi} \text{ implies } \tau(\Gamma) \models_{\mathcal{A}} \tau(\bar{\varphi}).$$

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Theorem

If τ *interprets* $\mathcal{L}$ *then* $\models_{\mathrm{Mod}^\tau(\mathcal{L})}$ *is the largest* τ*-interpretation of* $\mathcal{L}$.

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If τ *interprets* $\mathcal{L}$ *then* $\models_{\mathrm{Mod}^\tau(\mathcal{L})}$ *is the largest τ-interpretation of $\mathcal{L}$.*

Theorem

Let τ *be a translation that commutes with substitutions. Then if* $\vdash_{\mathcal{L}}$ *is axiomatized by* Φ *then* $\models_{\mathrm{Mod}^\tau(\mathcal{L})}$ *is axiomatized by* $\tau(\Phi)$*.*

(Generalized) refinements by translation

Definition (Refinement via interpretation)

Let $\tau : \mathrm{Fm}^k_\Sigma(\mathrm{Va}) \to \mathcal{P}(\mathrm{Fm}^m_{\Sigma'}(\mathrm{Va}))$ *be an interpretation of* $\mathcal{L}$*.* $\mathcal{L} \rightharpoonup_\tau \mathcal{L}'$*, if for any* $\Gamma \cup \{\bar{\varphi}\} \subseteq \mathrm{Fm}^k_\Sigma(\mathrm{Va})$,

$$\Gamma \vdash_{\mathcal{L}} \bar{\varphi} \;\Rightarrow\; \tau(\Gamma) \vdash_{\mathcal{L}'} \tau(\bar{\varphi}).$$

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$$\Gamma \vdash_{\mathcal{L}} \bar{\varphi} \ \Rightarrow \ \tau(\Gamma) \vdash_{\mathcal{L}'} \tau(\bar{\varphi}).$$

Example

I *-*

$$\mathrm{CPC} \rightharpoondown_{id} \mathrm{K}$$

since K is obtained from CPC *by adding* $\Box$ *to the signature, the axiom* $\Box(p \to q) \to (\Box p \to \Box q)$ *and the inference rule* $\dfrac{p}{\Box p}$,

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Let $\tau : \mathrm{Fm}^k_\Sigma(\mathrm{Va}) \to \mathcal{P}(\mathrm{Fm}^m_{\Sigma'}(\mathrm{Va}))$ *be an interpretation of* $\mathcal{L}$*.* $\mathcal{L} \rightharpoonup_\tau \mathcal{L}'$*, if for any* $\Gamma \cup \{\bar{\varphi}\} \subseteq \mathrm{Fm}^k_\Sigma(\mathrm{Va})$,

$$\Gamma \vdash_{\mathcal{L}} \bar{\varphi} \;\Rightarrow\; \tau(\Gamma) \vdash_{\mathcal{L}'} \tau(\bar{\varphi}).$$

Example

I -

$$\mathrm{CPC} \rightharpoonup_{id} \mathrm{K}$$

since K *is obtained from* CPC *by adding* $\Box$ *to the signature, the axiom* $\Box(p \to q) \to (\Box p \to \Box q)$ *and the inference rule* $\dfrac{p}{\Box p}$,

II-

$$\mathrm{CPC} \rightharpoonup_\tau \mathbb{HA} \rightharpoonup_\rho \mathrm{IPC}$$

where $\tau(p) = \{\langle \neg\neg p, \top \rangle\}$ *and* $\rho(\langle p, q \rangle) = \{p \to q, q \to p\}$.

Behavioral specification

Principle

The satisfaction of the requirements does not need to be strict, and may be checked up to a behavioral relation.

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- Data encapsulation is very important, for security reasons AND to allow effective and fast software updates.

Observational signature

Let $\Sigma = \langle S, \Omega \rangle$ and $\mathrm{Obs} \subseteq S$, *the observational signature Σ w.r.t Obs* is the pair $\langle \Sigma, \mathrm{Obs} \rangle$. The sorts Obs are called *observable sorts*.

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Example

```
Gen
    elt;
    cell;
Obs
    elt;
Op
    put: elt,cell -> cell;
    get:cell -> elt;
Ax

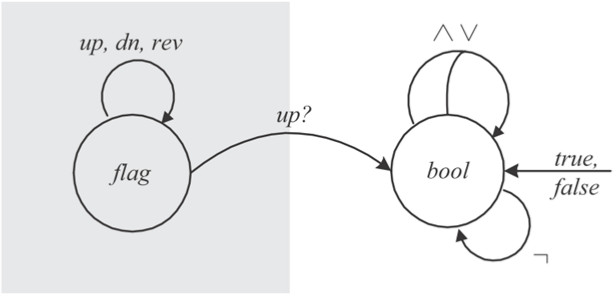
get(put(e,c))=e;
```

Example

```
Spec FLAGS = enrich BA by
Gen
    flag;
Obs
    bool;
Op
    up: flag -> flag;
    dn: flag -> flag;
    rev: flag -> flag;
    up?: flag -> bool;
```

```
Ax

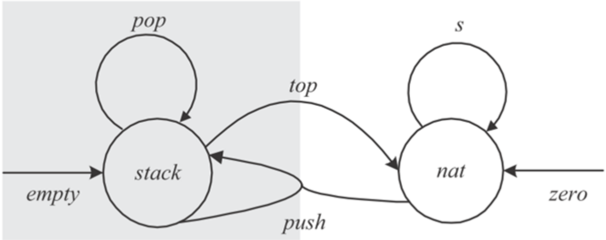
up?(up(x))=true;
up?(dn(x))=false;
up?(rev(x))=¬(up?(x));
```



Example

```
Spec STACK = = enrich Nat by
Gen
    stack;
Obs
    nat;
Op
    push:nat,stack -> stack;
    pop:stack -> stack;
    top:stack -> nat;
Ax

pop(push(x,s))=s;
top(push(x,s))=x;
```



Observational equality

Definition (Contexts and Observable Contexts)

Let $\langle \Sigma, \mathrm{Obs} \rangle$ *be an observational signature,* $X = (X_s)_{s \in S}$ *a family of infinite countable sets of variables (pairwise disjoint) and* $Z = \langle \{z_s\} \rangle_{s \in S}$ *an* S*-singular family of sets (pairwise disjoint) of different variables from the variables in* X*. pausa An s-context over* Σ *is a term* $c \in T(\Sigma, X \cup \{z_s\})_{s'}$*, for some* $s' \in S$*, with at least one occurrence of the variable* z_s*.*

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If $s' \in \mathrm{Obs}$, c is called an s-observable context.

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Let $\langle \Sigma, \mathrm{Obs} \rangle$ be an observational signature and $\mathbf{A}$ a Σ-algebra. $a, a' \in A_s$ are observationally equal w.r.t. Obs, $a \equiv_{\mathbf{A}}^{\mathrm{Obs}} a'$, if for any observable s-context $c(x_1 : s_1, \ldots, x_n : s_n, z_s)$, and every $b_1 \in A_{S_1}, \ldots, b_n \in A_{S_n}$,

$$c^{\mathbf{A}}(b_1, \ldots, b_n, a) = c^{\mathbf{A}}(b_1, \ldots, b_n, a').$$

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Fact

$\equiv_{\mathbf{A}}^{\mathrm{Obs}}$ is a congruence on $\mathbf{A}$.

Behavioral satisfaction

Definition

Let $\langle \Sigma, \mathrm{Obs} \rangle$ *be an observational signature,* $\mathbf{A}$ *is a* Σ*-algebra and* $t, t' \in T(\Sigma, X)_s$. $\mathbf{A}$ *é is behavioral model of* $t \approx t'$, $\mathbf{A} \models^{\mathrm{Obs}} t \approx t'$, *if for any observable s-context* $c(x_1 : s_1, \ldots, x_n : s_n, z_s)$ $\mathbf{A} \models c[t] \approx c[t']$.

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Theorem

1. $\mathbf{A} \models^{\mathrm{Obs}} t \approx t'$ *iff* $\mathbf{A}/\equiv^{\mathrm{Obs}}_{\mathbf{A}} \models t \approx t'$;
2. $SP \models^{\mathrm{Obs}} t \approx t'$ *iff* $SP^{\mathrm{Obs}} \models t \approx t'$;
3. $Th^{\mathrm{Obs}}(\mathcal{C}) = Th(\mathcal{C}^{\mathrm{Obs}})$,

where $\mathcal{C}^{\mathrm{Obs}} = \{\mathbf{A}/\equiv^{\mathrm{Obs}}_{\mathbf{A}} | \mathbf{A} \in \mathcal{C}\}$ *and* $SP^{\mathrm{Obs}} = \mathrm{Mod}(SP)^{\mathrm{Obs}}$.

Coinduction

Theorem

$\equiv_{\mathbf{A}}^{\mathrm{Obs}}$ *is the largest congruence on* $\mathbf{A}$ *which is the identity in* A_{Obs}. *I.e., if* $\approx$ *is a congruence s.t.* $(\approx)_{\mathrm{Obs}} = \Delta_{A_{\mathrm{Obs}}}$ *(called hidden congruence), then* $\approx \subseteq \equiv_{\mathbf{A}}^{\mathrm{Obs}}$.

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1. Define an appropriated binary relation R on A;
2. Show that R is an hidden congruence;
3. Show that $a\ R\ a'$.

Example

In $\mathcal{L}_{Flags}$ we have that $rev^{\mathbf{A}}(rev^{\mathbf{A}}(a)) \equiv^{\mathbf{A}}_{\mathrm{Obs}} a$. However, $rev(rev(x)) \approx x$ is not an equational consequence of the specification $\mathcal{L}_{Flags}$.

An example

```
bth SET[X :: TRIV] is sort Set .
  op empty : -> Set .
  op _in_  : Elt Set -> Bool .
  op add : Elt Set -> Set .
  ops (_U_) (_&_) : Set Set -> Set .
  vars E E' : Elt .  vars S S' : Set .
  eq E in empty = false .
  eq E in add(E' , S) = (E == E') or (E in S).
  eq E in S U S' = (E in S)  or (E in S') .
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However it is behavioral valid. Use the following relation

$$S\ R\ S \quad iff \quad \forall e \quad e \text{ in} S\ iff\ e \text{ in } S'$$

Complementary topics

Related issues:

- Behavioral refinement

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- Calculus for structured behavioral specifications