Algebraic and Coalgebraic methods in software development

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Outline

1. Equational specification
   - Term algebra, free algebra, initial and final objects.
   - Equational calculus. Initial models.
   - Term rewriting
   - Generalizations

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Term Algebra

Definition (term)

Let Σ *be a signature and* $X = \langle X_s \rangle_{s \in S}$ *a* S*-sorted set of variables for* Σ. *The* S*-set* Σ*-terms over* X *is the smallest* S*-set* $\mathrm{T}(\Sigma, \mathrm{X})$ *s.t.:*

- $X_s \subseteq \mathrm{T}(\Sigma, \mathrm{X})_s$;
- $\Omega_{\epsilon,s} \subseteq \mathrm{T}(\Sigma, \mathrm{X})_s$;
- *For any* $f : s_1, \ldots, s_n \to s \in \Sigma$ *and* $t_1 \in \mathrm{T}(\Sigma, \mathrm{X})_{s_1}, \ldots, t_n \in \mathrm{T}(\Sigma, \mathrm{X})_{s_n}$, $f(t_1, \ldots, t_n) \in \mathrm{T}(\Sigma, \mathrm{X})_s$;

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Definition (Term Algebra)

If $\mathrm{T}(\Sigma, \mathrm{X})$ is non empty, the term algebra over X is the algebra $\mathcal{T}(\Sigma, X)$ with carrier set $\mathrm{T}(\Sigma, \mathrm{X})$, and for any $f : s_1, \ldots, s_n \to s \in \Sigma$ and every $t_1 \in \mathrm{T}(\Sigma, \mathrm{X})_{s_1}, \ldots, t_n \in \mathrm{T}(\Sigma, \mathrm{X})_{s_n}$,

$$f^{\mathcal{T}(\Sigma, X)}(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$$

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Example (naturals revisited)

Since Σ_N is non empty, the term algebra exists. The carrier set is

$$0, s(0), s(s(0)), s(s(s(0))), \ldots$$

Example (A simple programming language)

```
Gen
    E
    bool
    P
Op
            0, x1, ..., xn :       → E
                     s, p : E → E
                 +, −, ∗ : E, E → E
                   _ = _ : E, E → bool
                  _ := _ : E, E → P
                    _;_ : P, P → P
if _ then _ else - fi : bool, P, P → P
     repeat _ do _ od : E, P → P
```

E correct expressions (for simplicity integers)
bool for booleans
P for programmes

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$$0, x_1, \ldots, x_n : \ \to \mathrm{E}$$
$$\mathrm{s}, \mathrm{p} : \mathrm{E} \to \mathrm{E}$$
$$+, -, * : \mathrm{E}, \mathrm{E} \to \mathrm{E}$$
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$$\mathtt{if}\ \_\ \mathtt{then}\ \_\ \mathtt{else}\ -\ \mathtt{fi} : \mathtt{bool}, \mathrm{P}, \mathrm{P} \to \mathrm{P}$$
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What means the following term?

$y_1 := 1; y_2 := 1;$
repeat 5 do
$\quad y_1 := y_1 * y_2;$
$\quad y_2 := y_2 + 1$
od

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Fact

For any signature Σ the trivial algebra is final in $Alg(\Sigma)$.

Example

I- The class of algebras over the signature of natural numbers $\Sigma = \{0, suc, +\}$, with just one sort *nat*, satisfying the axioms $suc(0 + n) = n$ and $suc(n) + m = suc(n + m)$ has both initial and final algebras.

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II- Moore Automata. Let IN and OUT be fixed. There is final algebra but not initial.

Gen

in
out
stat

Op

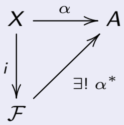
$c :\to inc \in In$
$k :\to outk \in Out$
$next : in, stat \to stat$
$print : stat \to out$

Show that there is no initial algebra but there is an interesting final algebra.

Algebra livre

Definition

Let K be a class of Σ-algebra. An algebra $\mathcal{F}$ (not necessarily in K) s.t. $X \subseteq F$ is called free for K over X iff for any $\mathcal{A} \in K$ and every $\alpha : X \to A$ there is a unique homomorphism $\alpha^* : \mathcal{F} \to \mathcal{A}$ that extends α, i.e., $\alpha^*(x) = \alpha(x)$ for all $x \in X$.



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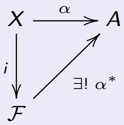

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Fact

If $\mathrm{T}(\Sigma, \mathrm{X})$ is non empty, $\mathcal{T}(\Sigma, X)$ is free in $Alg(\Sigma)$ over X.

Models and equations

- A Σ-equation is a pair $\langle t_1, t_2 \rangle$ with $t_1, t_2 \in \mathrm{T}(\Sigma, \mathrm{X})_s$. We will write $t_1 \approx t_2$.

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- $K \models t_1 \approx t_2$ if, for every $\mathcal{A} \in K$ $\mathcal{A} \models t_1 \approx t_2$.
- A pair $\langle \Sigma, \Phi \rangle$ is called a *flat specification*.

Galois connection

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Galois connection.

1. $\Phi \subseteq \Psi$ implies $\mathrm{Mod}(\Phi) \supseteq \mathrm{Mod}(\Psi)$;
2. $K \subseteq K'$ implies $\mathrm{Th}_{\Sigma}(K) \supseteq \mathrm{Th}_{\Sigma}(K')$;
3. $\Phi \subseteq \mathrm{Th}_{\Sigma}(\mathrm{Mod}(\Phi))$ and $K \subseteq \mathrm{Mod}(\mathrm{Th}_{\Sigma}(K))$.

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(vi) $\dfrac{\Phi \vdash_\Sigma t_1 \approx t_2}{\Phi \vdash_\Sigma \sigma(t_1) \approx \sigma(t_2)}$,for any substitution $\sigma : \mathrm{T}(\Sigma, \mathrm{X}) \to \mathrm{T}(\Sigma, \mathrm{X})$ (replacement)

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Examples

- Let $\Sigma = \langle S, \Omega \rangle$ with $S = \{S_0, S_1, S_2\}$, and Ω with $\Omega_{\epsilon, S_1} = \{a, b\}$, $\Omega_{\epsilon, S_2} = \{c, d\}$ and $\Omega_{S_1 S_2, S_0} = \{f\}$. Let $\Phi = \{a \approx b, c \approx d\}$. We have

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- [Flags] Let
$\Phi =$ Axioms of booleans $+ \{up?(dn(F)) \approx false, up?(up(F)) \approx true, up?(rev(F)) \approx \neg up?(F)\}$.
$\Phi \vdash rev(rev(F)) \approx F$?

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Show that $\Phi \vdash s(0) + n \approx s(n)$

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Proof.

($\Rightarrow$) Induction.
($\Leftarrow$) It is enough to show that $\Phi \models t_1 \approx t_2$ implies $\mathcal{T}(\Sigma, X)/ \equiv_\Phi \models t_1 \approx t_2$. □

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Theorem

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- Let $\mathcal{A} \in \mathrm{Mod}(\Phi)$. We say that $\mathcal{A}$ *contains junk* if it is not reachable and we say that $\mathcal{A}$ *contains confusion* if it satisfies a ground equation $t_1 \approx t_2 \in \mathrm{Eq}(\Sigma)$ s.t. $\Phi \not\models t_1 \approx t_2$.

Theorem

$\mathcal{T}(\Sigma)/\equiv_\Phi$ *is a model in* $\mathrm{Mod}(\Phi)$ *containing no junk and no confusion.*

Theorem

$\mathcal{T}(\Sigma)/\equiv_\Phi$ *is initial in* $\mathrm{Mod}(\Phi)$.

Corollary

Let $t_1 \approx t_2 \in \mathrm{Eq}(\Sigma)$, *i.e. ground equation. Then*

$$\mathcal{T}(\Sigma)/\equiv_\Phi \models t_1 \approx t_2 \Leftrightarrow \Phi \models t_1 \approx t_2.$$

[Bool]:

$\mathtt{bool}$

$true: \ \to \mathtt{bool}$
$false: \ \to \mathtt{bool}$
$\neg : \mathtt{bool} \to \mathtt{bool}$
$\wedge : \mathtt{bool},\mathtt{bool} \to \mathtt{bool}$
$\Rightarrow: \mathtt{bool},\mathtt{bool} \to \mathtt{bool}$

$\neg true \approx false$
$\neg false \approx true$
$p \wedge true \approx p$
$p \wedge false \approx false$
$p \wedge \neg p \approx false$
$p \Rightarrow q \approx \neg(p \wedge \neg q)$

(i) Present 3 finite models with 1, 2 and 3 elements.
(ii) Classify the models with respect to "junk" and "confusion".
(iii) Build the algebra $\mathcal{T}(\Sigma_{Bool})/\equiv_\Phi$, where Φ is the set of equations of the specification.

Term rewriting I

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Definition (Rewriting)

Let $t_1, t_2 \in T(\Sigma, X)_s$ and $r = u_1 \rhd u_2$ a rewriting rule over Σ. We say that t_1 directly reduces into t_2 by r, we write $t_1 \rhd_r t_2$, if there is a substitution $\alpha : X \to \mathrm{T}(\Sigma, \mathrm{X})$ s.t.:

- *$\alpha(u_1)$ is a subterm of t_1 and*
- *t_2 can be obtained from t_1 by replacing the subterm $\alpha(u_1)$ by $\alpha(u_2)$.*

- $\rhd_r$ is a binary relation over $T(\Sigma, X)$.

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- $\rhd_R = \bigcup_{r \in R} \rhd_r$.
- A computation is a sequence $t_1, \ldots, t_n \in \mathrm{T}(\Sigma, \mathrm{X})$ s.t. $t = t_1 \rhd_R \cdots \rhd_R t_n = t'$ and we write $t \rhd_R^* t'$ (it is the transitive closure of $\rhd_R$.).

Term rewriting II

Definition (Normal form)

Let $t, t' \in T(\Sigma, X)_s$ and R a rewriting system over Σ. t' is a normal form of t, we write $t \blacktriangleright_R t'$, if there is a terminating computation $t_1, \ldots, t_n$ s.t. $t = t_1$ and $t' = t_n$.

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In such case, we say that $t_1 \approx t_2$ can be deduced by rewriting in R, in symbols $\Vdash_R t_1 \approx t_2$, if there is a term t_3 s.t. $t_1 \blacktriangleright_R t_3$ and $t_2 \blacktriangleright_R t_3$.

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Theorem

If R is terminating and confluent then

$$\mathrm{Eq}(R) \vdash t_1 \approx t_2 \Rightarrow \Vdash_R t_1 \approx t_2.$$

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Example

Let $\Sigma = \langle \{S\}, \Omega \rangle$ where $\Omega_{\epsilon,S} = \{a, b\}$. Suppose that we would like to specify, using equations, the class of all Σ-algebras with exactly two elements. Birkhoff's theorem states that it can not be done.

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Example

Let $\Sigma = \langle \{S\}, \Omega \rangle$ where $\Omega_{\epsilon,S} = \{0\}$ and $\Omega_{S,S} = \{\times\}$.
The class K of Σ-algebras satisfying the familiar cancellation law: if $a \neq 0$ and $a \times b = a \times c$ then $b = c$, is not a variety.

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- More abstract - INSTITUTIONS.

Where is the Category Theory in this Module?

- Classes of algebras with respective morphisms defines a category.
  - Exercise – prove the validity of the category axioms
- A category of specifications can be naturally defined.
  - Exercise – define a suitable notion of specifications morphism
- The quotient construction is functorial
  - Exercise – show it
- ...

Where is the Category Theory in this Module?

Algebra categorically – to be revisited in the next module

- notion of algebra
- derivation of a polinomial functor F_Σ from an one-sorted algebraic signature Σ

An example

Any model of the signature

Sorts $account$

Ops $new :\to account$

$undo : account \to account$

$deposit : account \times \mathbb{Z} \to account$

$debit : account \times \mathbb{Z} \to account$

is an algebra

$$1 + X + X \times \mathbb{Z} + X \times \mathbb{Z} \xrightarrow{[undo, deposit, debit]} X$$