Taming Selective Strictness

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The Polymorphic Function *foldl*

$$\mathit{foldl} :: (\alpha \to \beta \to \alpha) \to \alpha \to [\beta] \to \alpha$$
$$\mathit{foldl}\ k\ z\ [\,] = z$$
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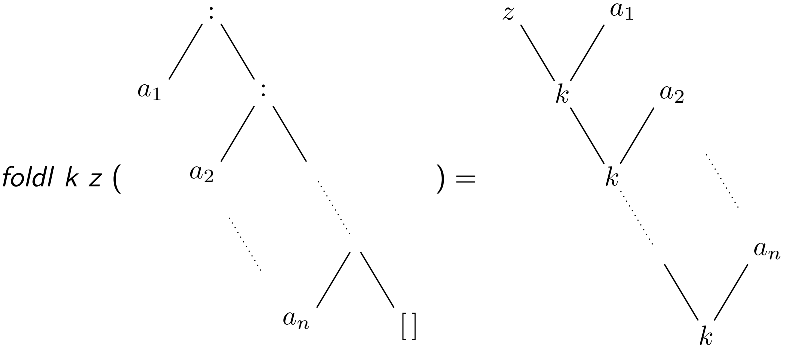

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$$\mathit{sum} = \mathit{foldl}\ (+)\ 0$$

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&= \mathit{foldl}\ (+)\ (((0+1)+2)+3)\ [\,] \\
&= (((0+1)+2)+3) \\
&= 6
\end{aligned}
$$

The Fusion Property

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$$(+1) \circ \mathit{sum}$$

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More generally:

$$f \circ (\mathit{foldl}\ k\ z) = \mathit{foldl}\ k'\ z'$$

For an inductive proof the conditions

$$f\ z = z'$$

$$\forall x, y.\ f\ (k\ x\ y) = k'\ (f\ x)\ y$$

are sufficient.

Free Theorems [Wadler, 1989]

With free theorems we can prove the fusion property automatically only using *foldl*'s type

[2] `http://www-ps.iai.uni-bonn.de/ft/`

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Output of a generator[2] for *foldl*'s type as input:

```
forall t1,t2 in TYPES,  f :: t1 -> t2 .
 forall t3,t4 in TYPES, g :: t3 -> t4.
  forall  k :: t1 -> t3 -> t1 .
   forall  k' :: t2 -> t4 -> t2 .
    (forall x :: t1. forall y :: t3.
        f (k x y) = k' (f x) (g y))
    ==> (forall z :: t1.
          forall xs :: [t3].
           f (foldl k  z  xs)
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Speed Up with Selective Strictness

Example (*sum* reconsidered)

$$\begin{aligned} \textit{sum}\ [1,2,3] &= \textit{foldl}\ (+)\ 0\ [1,2,3] \\ &= \textit{foldl}\ (+)\ (0+1)\ [2,3] \\ &= \textit{foldl}\ (+)\ ((0+1)+2)\ [3] \\ &\dots \end{aligned}$$

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Lazy evaluation results in a huge overhead.

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Haskell provides strict evaluation by the function *seq* $:: \alpha \to \beta \to \beta$:

$$\mathit{seq}\ a\ b = \begin{cases} b & \text{if } a \neq \bot \\ \bot & \text{otherwise} \end{cases}$$

foldl' — A Strict Version of foldl

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\begin{aligned}
&\mathit{foldl}' :: (\alpha \to \beta \to \alpha) \to \alpha \to [\beta] \to \alpha \\
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$$\mathit{sum}'\ [1,2,3] = \mathit{foldl}'\ (+)\ 0\ [1,2,3]$$

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\mathit{sum}'\ [1, 2, 3] &= \mathit{foldl}'\ (+)\ 0\ [1, 2, 3] \\
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sum' evaluates the addition whenever possible.

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$\Rightarrow$ Saving space (and time)

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\end{aligned}$$

sum' evaluates the addition whenever possible.

⇒ Saving space (and time)

⇒ Strict evaluation pays off here.

Drawbacks of Selective Strictness

Question:

$$f\ (foldl'\ k\ z\ xs) \stackrel{?}{=} foldl'\ k'\ (f\ z)\ xs$$

$$\text{if } f\ (k\ x\ y) = k'\ (f\ x)\ y$$

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Consider an example instantiation:

$$\begin{aligned} f &= \lambda x \rightarrow x \vee \bot \\ k = k' &= \lambda x\ y \rightarrow y \vee x \\ xs &= [False, True] \\ z &= False \end{aligned}$$

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$$f\ (foldl'\ k\ False\ [False, True]) = True$$

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$$f\ (foldl'\ k\ False\ [False, True]) = True$$

$$\neq$$

$$foldl'\ k'\ (f\ False)\ [False, True] = \bot$$

Drawbacks of Selective Strictness

Question:

$$f\ (foldl'\ k\ z\ xs)\ \boxed{\neq}\ foldl'\ k'\ (f\ z)\ xs$$
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Consider an example instantiation:

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Answer: No!

$$\begin{gathered} f\ (foldl'\ k\ False\ [False, True]) = True \\ \neq \\ foldl'\ k'\ (f\ False)\ [False, True] = \bot \end{gathered}$$

Analyzing the Problem

The strictness-aware free theorem:

```
forall t1,t2 in TYPES, f :: t1 -> t2, f strict and total.
 forall t3,t4 in TYPES, g :: t3 -> t4, g strict and total.
  forall k :: t1 -> t3 -> t1.
   forall k' :: t2 -> t4 -> t2.
    (((k /= _|_) <=> (k' /= _|_))
     && (forall x :: t1.
          ((k x /= _|_) <=> (k' (f x) /= _|_))
          && (forall y :: t3. f (k x y) = k' (f x) (g y))))
    ==> (forall z :: t1.
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    (( (k /= _|_) <=> (k' /= _|_) )
     && (forall x :: t1.
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Question: Are all these restrictions necessary?

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Question: Are all these restrictions necessary?

An inductive proof for

$$f\ (foldl'\ k\ z\ xs) = foldl'\ k'\ (f\ z)\ xs$$

shows that $f\ x = \perp \Leftrightarrow x = \perp$ suffices (i.e. f is strict and total).

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Solution: Make the use of *seq* visible from the type. In particular where it is used.

A Refined Type System . . .

Add new type constructors.

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was not sufficient [Johann and Voigtländer, 2004].

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. . . and its Effects on the Typing Rules (1)

A rule system for $\Gamma \vdash \tau \in \mathsf{Seqable}$:

$$\Gamma \vdash [\tau] \in \mathsf{Seqable} \qquad \Gamma \vdash (\tau_1 \rightarrow^{\varepsilon} \tau_2) \in \mathsf{Seqable}$$

$$\frac{\alpha^{\varepsilon} \in \Gamma}{\Gamma \vdash \alpha \in \mathsf{Seqable}} \qquad \frac{\alpha^{\varepsilon}, \Gamma \vdash \tau \in \mathsf{Seqable}}{\Gamma \vdash (\forall \alpha^{\nu}.\tau) \in \mathsf{Seqable}}$$

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with $\Gamma = \alpha_1^{\nu_1}, \ldots \alpha_n^{\nu_n}, x_1 :: \tau_1, \ldots x_n :: \tau_n$ and $\nu_i \in \{\circ, \varepsilon\}$.

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More typing rules because of new constructors:

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$$\frac{\Gamma \vdash t_1 :: \tau_1 \to^{\circ} \tau_2 \qquad \Gamma \vdash t_2 :: \tau_1}{\Gamma \vdash (t_1\ t_2) :: \tau_2}$$

A term can have more than one type.

$$(\lambda x :: \mathit{Int}.\, x) :: \mathit{Int} \to^{\varepsilon} \mathit{Int}$$
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...and its Effects on the Typing Rules (2)

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We introduce subtyping.

$$\frac{\Gamma \vdash t :: \tau_1 \qquad \tau_1 \preceq \tau_2}{\Gamma \vdash t :: \tau_2}\ (\textsc{Sub})$$

Refinement Pays Off

The use of selective strictness becomes visible from the type ($\circ$- and ε-marks):

$$\begin{array}{r}
\mathit{foldl} :: \forall^{\circ}\alpha.\forall^{\circ}\beta.(\alpha \to^{\circ} \beta \to^{\circ} \alpha) \to^{\varepsilon} \alpha \to^{\varepsilon} [\beta] \to^{\varepsilon} \alpha \\
\mathit{foldl}' :: \forall^{\varepsilon}\alpha.\forall^{\circ}\beta.(\alpha \to^{\circ} \beta \to^{\circ} \alpha) \to^{\varepsilon} \alpha \to^{\varepsilon} [\beta] \to^{\varepsilon} \alpha
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```
forall t1,t2 in TYPES, f :: t1 -> t2, f  strict and total .
  forall k :: t1 -> t3 -> t1.
   forall k' :: t2 -> t3 -> t2.
    (( (k /= _|_) <=> (k' /= _|_) )
     && (forall x :: t1.
          ( (k x /= _|_) <=> (k' (f x) /= _|_) )
          && (forall y :: t3. f (k x y) = k' (f x) y)))
    ==> (forall z :: t1.
          forall xs :: [t3].
            f ( foldl  k z xs) = foldl  k' (f z) xs)
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Restrictions on free theorems can be dropped if the type guarantees selective strictness is not used.

```
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Going Algorithmic

Goal: An algorithm retyping from standard types to (minimal) refined types.

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How to Deal with Refined Type Annotations?

Switch to variable marks at the type annotations:

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$\Downarrow$ combine two rules into one

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$\Downarrow$ add constraints for the mark variables

$$\frac{\langle \dot{\Gamma} \vdash \dot{t}_1 \rangle \Rrightarrow (C_1, \dot{\tau}_1 \to^{\nu} \dot{\tau}_2) \quad \langle \dot{\Gamma} \vdash \dot{t}_2 \rangle \Rrightarrow (C_2, \dot{\tau}_1') \quad \langle \dot{\tau}_1 = \dot{\tau}_1' \rangle \Rrightarrow C_3}{\langle \dot{\Gamma} \vdash \dot{t}_1\ \dot{t}_2 \rangle \Rrightarrow (C_1 \wedge C_2 \wedge C_3, \dot{\tau}_2)}$$

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How to get back to concrete types, without mark variables?

Back to Concrete Typability (Example)

We have:

$$\begin{array}{l}\langle \vdash \Lambda\alpha.\, \Lambda\beta.\, \lambda f :: \alpha \rightarrow^{\nu_1} \beta.\, \lambda x :: \alpha.\, \textbf{let}!\ x' = x \textbf{ in } f\ x' \rangle \Rrightarrow \\ ((\nu_2 = \varepsilon) \wedge (\nu_4 \leqslant \nu_1) \wedge (\nu_1 \leqslant \nu_6), \\ \qquad\qquad\qquad \forall^{\nu_2}\alpha.\, \forall^{\nu_3}\beta.\, (\alpha \rightarrow^{\nu_6} \beta) \rightarrow^{\nu_7} \alpha \rightarrow^{\nu_5} \beta)\end{array}$$

with $\circ < \varepsilon$.

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$$((\nu_2 = \varepsilon) \wedge (\nu_4 \leqslant \nu_1) \wedge (\nu_1 \leqslant \nu_6),$$
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We test all possible instantiations for the variable marks.

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$$\begin{aligned}
\nu_1 &= \circ \\
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$$(\mathit{False},$$
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Back to Concrete Typability (Example)

We have:

$$\langle \vdash \Lambda\alpha.\,\Lambda\beta.\,\lambda f :: \alpha \rightarrow^{\circ} \beta.\,\lambda x :: \alpha.\,\mathbf{let!}\ x' = x\ \mathbf{in}\ f\ x' \rangle \Rrightarrow$$
$$(\mathit{True},$$
$$\forall^{\varepsilon}\alpha.\,\forall^{\circ}\beta.\,(\alpha \rightarrow^{\circ} \beta) \rightarrow^{\varepsilon} \alpha \rightarrow^{\varepsilon} \beta)$$

with $\circ < \varepsilon$.

We test all possible instantiations for the variable marks.

$$\begin{array}{lcl@{\qquad}lcl}
\nu_1 & = & \varepsilon & \nu_1 & = & \circ \\
\nu_2 & = & \varepsilon & \nu_2 & = & \varepsilon \\
\nu_3 & = & \varepsilon & \nu_3 & = & \circ \\
\nu_4 & = & \varepsilon & \nu_4 & = & \circ \\
\nu_5 & = & \varepsilon & \nu_5 & = & \varepsilon \\
\nu_6 & = & \varepsilon & \nu_6 & = & \circ \\
\nu_7 & = & \varepsilon & \nu_7 & = & \varepsilon
\end{array}$$

Back to Concrete Typability (Example)

We have:

$$\langle \vdash \Lambda\alpha.\, \Lambda\beta.\, \lambda f :: \alpha \rightarrow^{\circ} \beta.\, \lambda x :: \alpha.\, \textbf{let}!\ x' = x\ \textbf{in}\ f\ x' \rangle \Rrightarrow$$
$$(\mathit{True},$$
$$\forall^{\varepsilon}\alpha.\, \forall^{\circ}\beta.\, (\alpha \rightarrow^{\circ} \beta) \rightarrow^{\varepsilon} \alpha \rightarrow^{\varepsilon} \beta)$$

with $\circ < \varepsilon$.

We test all possible instantiations for the variable marks.

$$\begin{aligned} \nu_1 &= \circ \\ \nu_2 &= \varepsilon \\ \nu_3 &= \circ \\ \nu_4 &= \circ \\ \nu_5 &= \varepsilon \\ \nu_6 &= \circ \\ \nu_7 &= \varepsilon \end{aligned}$$

We take only the minimal solution!

Make it a Type Refinement Algorithm

$\boxed{\text{input: closed term with standard type annotations}}$

Make it a Type Refinement Algorithm

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$\Downarrow$ add variable marks

$\boxed{\text{term with parameterized refined type annotations}}$

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Make it a Type Refinement Algorithm

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$\Downarrow$ the main algorithm

$$\boxed{\text{constraint and parameterized type}}$$

$\Downarrow$ solve constraint

$$\boxed{\text{all possible refined types}}$$

$\Downarrow$ type comparison

$$\boxed{\text{output: the refined types leading to the strongest free theorems}}$$

The Webinterface

The term

```
t = (/\a.
     (/\b.
      (\c::(a -> (b -> a)).
       (fix (\h::(a -> ([b] -> a)).
             (\n::a.
              (\ys::[b].
               (seq (c n) (case ys of {[] -> n; x:xs ->
                           (seq xs (seq x (let n' = ((c n) x) in
                                           ((h n') xs))))})))))))))
```

can be typed to the optimal type

```
(forall^n a. (forall^e b. ((a ->^n (b ->^e a)) ->^e (a ->^e ([b] ->^e a)))))
```

with the free theorem

```
forall t1,t2 in TYPES, f :: t1 -> t2, f strict.
 forall t3,t4 in TYPES, g :: t3 -> t4, g strict and total.
  ((t_{t1}_{t3} /= _|_) <=> (t_{t2}_{t4} /= _|_))
  && (forall p :: t1 -> (t3 -> t1).
       forall q :: t2 -> (t4 -> t2).
        (forall x :: t1.
          ((p x /= _|_) <=> (q (f x) /= _|_))
          && (forall y :: t3. f (p x y) = q (f x) (g y)))
        ==> (((t_{t1}_{t3} p /= _|_) <=> (t_{t2}_{t4} q /= _|_))
             && (forall z :: t1.
                  ((t_{t1}_{t3} p z /= _|_) <=> (t_{t2}_{t4} q (f z) /= _|_))
                  && (forall v :: [t3].
                       f (t_{t1}_{t3} p z v) = t_{t2}_{t4} q (f z) (map_{t3}_{t4} g v)))))
```

The normal free theorem for the type without marks would be:

http://www-ps.iai.uni-bonn.de/cgi-bin/polyseq.cgi

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