

# Practical Coinduction

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## Abstract

Induction is a well-established proof principle that is taught in most undergraduate programs in mathematics and computer science. In computer science, it is used primarily to reason about inductively-defined datatypes such as finite lists, finite trees, and the natural numbers. Coinduction is the dual principle that can be used to reason about coinductive datatypes such as infinite streams or trees, but it is not as widespread or as well understood. In this paper, we illustrate through several examples the use of coinduction in informal mathematical arguments. Our aim is to promote the principle as a useful tool for the working mathematician and to bring it to a level of familiarity on par with induction. We show that coinduction is not only about bisimilarity and equality of behaviors, but also applicable to a variety of functions and relations defined on coinductive datatypes.

## 1 Introduction

Perhaps the single most important general proof principle in computer science, and arguably in all of mathematics, is *induction*. There is a valid induction principle corresponding to any well-founded relation, but in computer science, it is most often seen in the form known as *structural induction*, in which the domain of discourse is an inductively-defined datatype such as finite lists, finite trees, or the natural numbers.

For example, consider the type List of  $A$  of finite lists over an alphabet  $A$ , defined inductively by

- $\text{nil} \in \text{List of } A$
- if  $a \in A$  and  $\ell \in \text{List of } A$ , then  $a :: \ell \in \text{List of } A$ .

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The defined datatype is the least solution of the equation

$$\text{List of } A = \text{nil} + A \times \text{List of } A. \quad (1.1)$$

It is the initial algebra for a signature consisting of one constant (nil) and one binary constructor (::). This means that one can define functions with domain List of  $A$  uniquely by structural induction. For example, the functions length, which computes the length of a finite list, and concat, which concatenates two finite lists, can be defined as follows.

$$\begin{aligned} \text{length}(\text{nil}) &= 0 & \text{concat}(\text{nil}, \ell) &= \ell \\ \text{length}(a :: \ell) &= 1 + \text{length}(\ell) & \text{concat}(a :: \ell_1, \ell_2) &= a :: \text{concat}(\ell_1, \ell_2). \end{aligned}$$

No one would dispute that these functions are uniquely defined. Now we can prove that  $\text{length}(\text{concat}(\ell_1, \ell_2)) = \text{length}(\ell_1) + \text{length}(\ell_2)$  by structural induction on the first argument.

$$\begin{aligned} \text{length}(\text{concat}(\text{nil}, \ell_2)) &= \text{length}(\ell_2) \\ &= 0 + \text{length}(\ell_2) = \text{length}(\text{nil}) + \text{length}(\ell_2) \\ \text{length}(\text{concat}(a :: \ell_1, \ell_2)) &= \text{length}(a :: \text{concat}(\ell_1, \ell_2)) \\ &= 1 + \text{length}(\text{concat}(\ell_1, \ell_2)) \\ &= 1 + \text{length}(\ell_1) + \text{length}(\ell_2) \quad (\text{inductive step}) \\ &= \text{length}(a :: \ell_1) + \text{length}(\ell_2) \end{aligned}$$

This proof would raise no objections as to its correctness. The induction principle in play here is implicit and trusted; there is no need to reassert its validity every time it is used or whenever a new inductive datatype is introduced.

Coinduction, on the other hand, is still mysterious and unfamiliar to many. Coinduction is the dual principle to induction and is used to prove properties of coinductively-defined datatypes such as infinite streams, infinite trees, and coterms. These datatypes are typically final coalgebras for a signature. For example, the finite and infinite streams over  $A$  form the final coalgebra for the signature (nil, ::) and are the greatest solution of (1.1).

Although coinduction has been around for decades, many proofs in the literature that rely on coinduction still end up essentially reasserting the principle every time it is used. It is clearly not as familiar as induction and not trusted in the same way. Quoting Rutten from his seminal paper on universal coalgebra:

*Firstly, induction principles are well known and much used. The coinductive definition and proof principles for coalgebras are less well known by far, and often even not very clearly formulated. [17]*

This paper was the precursor of much work on coalgebra and coinduction, which included, among many others, extensions to modal logics [12, 18, 19, 20]

and structural operational semantics [8, 21]. However, most attention has been devoted to bisimulation proofs of equality between coinductively-defined objects. With few exceptions [2, 4, 15], not much has been explored when it comes to properties of other relations on coinductive datatypes besides equality.

Our aim in this paper is to introduce an informal style of coinductive reasoning that can be quite useful in dealing with infinite data. We illustrate this style with a number of interesting examples. Our arguments may seem a bit magical at first, because they apply to infinite objects and look something like induction without a basis. Nevertheless, they are backed by a sound formal proof principle. The reason they work is summed up in the following motto:

*A property holds by induction if there is good reason for it to hold; whereas a property holds by coinduction if there is no good reason for it not to hold.*

Although there is a *coinductive step* but no basis, any difficulty that would arise that would cause the property not to hold would manifest itself in the attempt to prove the coinductive step.

The examples we give in the paper demonstrate the versatility of the principle. We will prove properties of several kinds:

- **Classical bisimulation proofs.** For example, given two coinductively-defined streams, are they equal?
- **Properties other than equality.** For example, given two streams  $\sigma$  and  $\tau$  over  $\mathbb{N}$ , is  $\sigma$  lexicographically less than  $\tau$ ?
- **Properties of relations on coinductive datatypes.** For example, is the subtype order on recursive types transitive?
- **Properties of functions between coinductive datatypes.** For example, given two coinductively-defined partial orders and a function between them, is the function monotone?

In all these examples, the proofs we give are quite short and involve establishing a *coinductive step* analogous to the inductive step in proofs by induction. What is missing is the final argument that the proof is a valid application of the coinduction principle; but it is not necessary to include this step for the same reason that it is not necessary to argue with every inductive proof that the proof is a valid application of the induction principle.

We hope that this paper will be of interest both to experts in coalgebra and coinduction by pointing out nonstandard examples of proofs by coinduction and to nonexperts by showing how coinduction can be used to prove interesting properties from the realm of functional and imperative programming.

## 2 Coinductive Datatypes

Coinductive datatypes usually refer to possibly infinite structures. Prime examples include infinite streams, infinite trees, coterms (infinite terms), and finite and infinite words over an alphabet. In programming language semantics, coinductive types are often used to model traces [5], recursive types [2], and program state [7].

Formally, coinductive datatypes can be defined as elements of a final coalgebra for a given polynomial endofunctor on  $\text{Set}$ . For instance, the set  $A^\omega$  of infinite streams over an alphabet  $A$  is (the carrier of) the final coalgebra of the functor  $FX = A \times X$ , whereas the set  $A^\infty$  of finite and infinite words over an alphabet  $A$  is the final coalgebra of  $FX = \mathbb{1} + A \times X$ .

Many functional programming languages such as Haskell and OCaml support coinductive types; Standard ML and F# do not. The type of streams in Haskell would be defined as

$$\text{data Stream } a = \text{S } a \text{ (Stream } a).$$

Coinductive datatypes are usually presented together with their *destructors*. For instance, streams admit two operations  $\text{hd} : A^\omega \rightarrow A$  and  $\text{tl} : A^\omega \rightarrow A^\omega$ , which in Haskell would be defined as

$$\text{hd (S } a \text{ as)} = a \qquad \text{tl (S } a \text{ as)} = \text{as}.$$

The existence of destructors is a consequence of the fact that  $A^\omega$  is a coalgebra for the functor  $FX = A \times X$ . All such coalgebras come equipped with a *structure map*  $\langle \text{obs}, \text{cont} \rangle : X \rightarrow A \times X$ ; for  $A^\omega$ ,  $\text{obs} = \text{hd}$  and  $\text{cont} = \text{tl}$ . Interestingly enough, the structure map of a final coalgebra is always an isomorphism, as is the structure map of an initial algebra. This is the content of *Lambek's lemma* [13]. Thus initial algebras and final coalgebras are always both algebras and coalgebras for the same functor. In the case of streams, the inverse of  $\langle \text{hd}, \text{tl} \rangle$ , usually referred to as the *constructor*, is  $:: (\text{cons})$ , a function of type  $A \times A^\omega \rightarrow A^\omega$ . In Haskell, it would be defined as

$$:: (a, \text{as}) = \text{S } a \text{ as}.$$

The  $A$ -streams  $A^\omega$  can also be characterized as the cofree coalgebra on generators  $A$  for the identity functor  $FX = X$ . This means that the functor  $A \mapsto (A^\omega, \text{tl})$  is right adjoint to the forgetful functor  $(X, \text{cont}) \mapsto X$ . In this characterization,  $\text{hd}$  is the counit of the adjunction.

## 3 Some Motivating Examples

### 3.1 Lexicographic Order on Streams

In this section, we give an informal proof that lexicographic order on streams is transitive. The argument illustrates the use of coinduction in a nonstandard setting. At first glance, this technique seems quite magical because it appears to involve induction on a non-well-founded relation.

Let  $(A, \leq)$  be a partially ordered alphabet. An  $A$ -stream is an element of  $A^\omega$ . The constructor  $::$  (cons) of type  $A \times A^\omega \rightarrow A^\omega$  and corresponding destructors  $\text{hd} : A^\omega \rightarrow A$  and  $\text{tl} : A^\omega \rightarrow A^\omega$  are defined as in §1. The ordering  $\leq_{\text{lex}}$  on  $A$ -streams is defined to be the *maximum* relation  $R \subseteq A^\omega \times A^\omega$  satisfying the following property:

**Property 3.1** *If  $\sigma R \tau$ , then*

- (i)  $\text{hd}(\sigma) \leq \text{hd}(\tau)$ , and
- (ii) *if  $\text{hd}(\sigma) = \text{hd}(\tau)$ , then  $\text{tl}(\sigma) R \text{tl}(\tau)$ .*

The relation  $\leq_{\text{lex}}$  exists and is unique, and any relation satisfying Property 3.1 is a subset. This is because if  $\{R_\alpha\}$  is any indexed family of relations satisfying Property 3.1, then their union  $\bigcup_\alpha R_\alpha$  also satisfies Property 3.1. The relation  $\leq_{\text{lex}}$  is thus the union of all relations satisfying Property 3.1.

The relation  $\leq_{\text{lex}}$  satisfies many desirable properties. For example,  $\leq_{\text{lex}}$  is reflexive, that is,  $\sigma \leq_{\text{lex}} \sigma$  holds for any  $A$ -stream  $\sigma$ , because the identity relation  $\text{id} = \{(\sigma, \sigma) \mid \sigma \in A^\omega\}$  satisfies Property 3.1, therefore  $\text{id} \subseteq \leq_{\text{lex}}$ .

Moreover, because  $\leq_{\text{lex}}$  is maximum, the converse of Property 3.1 holds for  $\leq_{\text{lex}}$ ; that is, if

- (i)  $\text{hd}(\sigma) \leq \text{hd}(\tau)$ , and
- (ii)  $\text{hd}(\sigma) = \text{hd}(\tau) \Rightarrow \text{tl}(\sigma) \leq_{\text{lex}} \text{tl}(\tau)$ ,

then  $\sigma \leq_{\text{lex}} \tau$ . If not, then  $\leq_{\text{lex}}$  would not be maximal; one could add the pair  $(\sigma, \tau)$  to  $\leq_{\text{lex}}$  without violating Property 3.1.

To say that  $\leq_{\text{lex}}$  is the maximum relation satisfying Property 3.1 says that it is the greatest fixpoint of the operator

$$T_{\leq_{\text{lex}}}(R) = \{(\sigma, \tau) \mid \text{hd}(\sigma) \leq \text{hd}(\tau) \text{ and } \text{hd}(\sigma) = \text{hd}(\tau) \Rightarrow \text{tl}(\sigma) R \text{tl}(\tau)\}.$$

Formally, the relation  $\leq_{\text{lex}}$  is defined as the greatest fixpoint of  $T_{\leq_{\text{lex}}}$ ; in symbols,  $\leq_{\text{lex}} = \nu X. T_{\leq_{\text{lex}}}(X)$ .

Now we will use coinduction to show

**Theorem 3.2** *The relation  $\leq_{\text{lex}}$  is transitive.*

*Proof.* We want to show that if  $\sigma \leq_{\text{lex}} \rho \leq_{\text{lex}} \tau$ , then  $\sigma \leq_{\text{lex}} \tau$ . Suppose

$$\sigma \leq_{\text{lex}} \rho \leq_{\text{lex}} \tau. \quad (3.2)$$

By Property 3.1(i),

$$\text{hd}(\sigma) \leq \text{hd}(\rho) \leq \text{hd}(\tau). \quad (3.3)$$

Since  $\leq$  is transitive on  $A$ ,  $\text{hd}(\sigma) \leq \text{hd}(\tau)$ . Thus Property 3.1(i) holds for the pair  $\sigma, \tau$ .

For Property 3.1(ii), if  $\text{hd}(\sigma) = \text{hd}(\tau)$ , then  $\text{hd}(\sigma) = \text{hd}(\rho) = \text{hd}(\tau)$  by (3.3) and the antisymmetry of  $\leq$  on  $A$ . By the assumption (3.2) and Property 3.1(ii),  $\text{tl}(\sigma) \leq_{\text{lex}} \text{tl}(\rho) \leq_{\text{lex}} \text{tl}(\tau)$ . By the coinduction hypothesis,  $\text{tl}(\sigma) \leq_{\text{lex}} \text{tl}(\tau)$ . This establishes Property 3.1(ii) for  $\sigma, \tau$ .

We have shown that under the assumption (3.2) and the coinduction hypotheses on the tails, both clauses (i) and (ii) of Property 3.1 hold for the pair  $\sigma, \tau$ . By the converse of Property 3.1,  $\sigma \leq_{\text{lex}} \tau$ .  $\square$

The part of this proof that is unsettling is the appeal to the coinduction hypothesis on the tails of the two streams. Streams are infinite, and there is nothing like a basis. So the entire argument seems non-well-founded. But as we will show later the argument is quite firmly grounded. Intuitively, one can appeal to the coinductive hypothesis as long as there has been progress in observing the elements of the stream (guardedness) and there is no further analysis of the tails (opacity). We will explain this formally in §4.

There are of course other ways to prove transitivity of  $\leq_{\text{lex}}$ . Here is an informal proof by induction that is dual to the proof presented above.

*Proof of Theorem 3.2 (alternative).* We show the contrapositive: For any  $\sigma, \rho, \tau$ , if  $\sigma \not\leq_{\text{lex}} \tau$ , then either  $\sigma \not\leq_{\text{lex}} \rho$  or  $\rho \not\leq_{\text{lex}} \tau$ . We proceed by induction on the inductive definition of  $\not\leq_{\text{lex}}$ .

If  $\sigma \not\leq_{\text{lex}} \tau$  because of (i), then  $\text{hd}(\sigma) \not\leq \text{hd}(\tau)$ , therefore either  $\text{hd}(\sigma) \not\leq \text{hd}(\rho)$  or  $\text{hd}(\rho) \not\leq \text{hd}(\tau)$ , since  $\leq$  is transitive on  $A$ . Then either  $\sigma \not\leq_{\text{lex}} \rho$  or  $\rho \not\leq_{\text{lex}} \tau$  by (i). This is the basis.

If  $\sigma \not\leq_{\text{lex}} \tau$  because of (ii), then  $\text{hd}(\sigma) = \text{hd}(\tau)$  and  $\text{tl}(\sigma) \not\leq_{\text{lex}} \text{tl}(\tau)$ , and  $\text{tl}(\sigma) \not\leq_{\text{lex}} \text{tl}(\tau)$  was determined at an earlier stage in the inductive definition. By the induction hypothesis, either  $\text{tl}(\sigma) \not\leq_{\text{lex}} \text{tl}(\rho)$  or  $\text{tl}(\rho) \not\leq_{\text{lex}} \text{tl}(\tau)$ , say the former without loss of generality. If either  $\text{hd}(\sigma) \not\leq \text{hd}(\rho)$  or  $\text{hd}(\rho) \not\leq \text{hd}(\tau)$ , we are done as above. Otherwise  $\text{hd}(\sigma) \leq \text{hd}(\rho) \leq \text{hd}(\tau)$ , and since  $\text{hd}(\sigma) = \text{hd}(\tau)$ , we have  $\text{hd}(\sigma) = \text{hd}(\rho) = \text{hd}(\tau)$ . Since  $\text{tl}(\sigma) \not\leq_{\text{lex}} \text{tl}(\rho)$ , we have  $\sigma \not\leq_{\text{lex}} \rho$  by (ii).  $\square$

In the latter proof, we are actually doing induction on the relation

$$\{((\text{tl}(\sigma), \text{tl}(\tau)), (\sigma, \tau)) \mid \text{hd}(\sigma) = \text{hd}(\tau)\},$$

which is well-founded on the set  $\not\leq_{\text{lex}}$ . One can show that  $\sigma \not\leq_{\text{lex}} \tau$  iff there exists  $n \geq 0$  such that  $\text{hd}(\text{tl}^n(\sigma)) = \text{hd}(\text{tl}^n(\tau))$  for  $m < n$  and  $\text{hd}(\text{tl}^n(\sigma)) \not\leq \text{hd}(\text{tl}^n(\tau))$ . The smallest such  $n$  is the stage in the inductive definition of  $\not\leq_{\text{lex}}$  at which  $\sigma \not\leq_{\text{lex}} \tau$  is established.

Here is another example involving lexicographic order on streams.

**Theorem 3.3** *For streams over a commutative semigroup  $(A, +)$ , pointwise addition is monotone; that is,*

$$\sigma \leq_{\text{lex}} \tau \text{ and } \rho \leq_{\text{lex}} \pi \Rightarrow \sigma + \rho \leq_{\text{lex}} \tau + \pi,$$

where  $\sigma + \tau$  is the pointwise sum of the two streams.

*Proof.* First observe that the pointwise sum operation  $+$  on streams satisfies the equations

$$\text{hd}(\sigma + \tau) = \text{hd}(\sigma) + \text{hd}(\tau) \quad \text{tl}(\sigma + \tau) = \text{tl}(\sigma) + \text{tl}(\tau). \quad (3.4)$$

By Property 3.1(i),

$$\text{hd}(\sigma + \rho) = \text{hd}(\sigma) + \text{hd}(\rho) \leq \text{hd}(\tau) + \text{hd}(\pi) = \text{hd}(\tau + \pi).$$

Thus, Property 3.1(i) holds for the pair  $(\sigma + \rho, \tau + \pi)$ .

For Property 3.1(ii), if  $\text{hd}(\sigma + \rho) = \text{hd}(\tau + \pi)$  and using the fact that, by hypothesis,  $\text{hd}(\sigma) \leq \text{hd}(\tau)$  and  $\text{hd}(\rho) \leq \text{hd}(\pi)$ , then we can conclude that  $\text{hd}(\sigma) = \text{hd}(\tau)$  and  $\text{hd}(\rho) = \text{hd}(\pi)$ . By the assumptions  $\sigma \leq_{\text{lex}} \tau$  and  $\rho \leq_{\text{lex}} \pi$  and Property 3.1(ii),  $\text{tl}(\sigma) \leq_{\text{lex}} \text{tl}(\tau)$  and  $\text{tl}(\rho) \leq_{\text{lex}} \text{tl}(\pi)$ . By the coinduction hypothesis, we have  $\text{tl}(\sigma) + \text{tl}(\rho) \leq_{\text{lex}} \text{tl}(\tau) + \text{tl}(\pi)$ . That is,  $\text{tl}(\sigma + \rho) \leq_{\text{lex}} \text{tl}(\tau + \pi)$ . This establishes Property 3.1(ii) for  $(\sigma + \rho, \tau + \pi)$ .  $\square$

A subtle but important observation is that the equations (3.4) determine the operation  $+$  on streams uniquely. Indeed, this would be the preferred way to *define* the operation  $+$  coinductively for the purpose of formalization in an automated deduction system such as Coq or NuPrl, as the informal definition above using pointwise sum would require the extraneous notions of the natural numbers and indexing.

But how do we know from (3.4) alone that  $+$  exists and is unique? Ultimately, this comes from the fact that  $(A^\omega, \text{hd}, \text{tl})$  is a final coalgebra. This means that for any coalgebra  $(X, \text{obs}, \text{cont})$  with  $\text{obs} : X \rightarrow A$  and  $\text{cont} : X \rightarrow X$ , there is a unique coalgebra morphism  $X \rightarrow A^\omega$ . If we make a coalgebra out of  $A^\omega \times A^\omega$  by defining

$$\text{obs}(\sigma, \tau) = \text{hd}(\sigma) + \text{hd}(\tau) \quad \text{cont}(\sigma, \tau) = (\text{tl}(\sigma), \text{tl}(\tau)),$$

then  $+$  is the unique morphism to the final coalgebra  $A^\omega$ , the equations (3.4) asserting exactly that  $+$  is a coalgebra morphism.

## 3.2 Recursive Types

Recursive types were introduced by Mendler [14]. The subtyping problem for recursive types was studied in [1, 2, 10]. In their simplest form, recursive types are constructed from the constants  $\perp$  and  $\top$  and the binary function space constructor  $\rightarrow$ . The set of *recursive types*  $C$  is the set of coterms of this signature. The subtype order  $\leq$  is defined to be the greatest binary relation on  $C$  such that if  $\sigma \leq \tau$ , then either

- $\sigma = \perp$ , or
- $\tau = \top$ , or
- $\sigma = \sigma_1 \rightarrow \sigma_2$  and  $\tau = \tau_1 \rightarrow \tau_2$  and  $\tau_1 \leq \sigma_1$  and  $\sigma_2 \leq \tau_2$ .

In other words,  $\leq$  is  $\nu X.T(X)$ , the greatest postfixpoint of the monotone map

$$T(X) = \{(\perp, \tau) \mid \tau \in C\} \cup \{(\sigma, \top) \mid \sigma \in C\} \\ \cup \{(\sigma_1 \rightarrow \sigma_2, \tau_1 \rightarrow \tau_2) \mid (\tau_1, \sigma_1) \in X, (\sigma_2, \tau_2) \in X\}.$$

**Theorem 3.4**  $\leq$  is transitive.

*Proof.* Suppose  $\sigma \leq \rho \leq \tau$ . If  $\sigma = \perp$  or  $\tau = \top$ , we are done. Otherwise, we cannot have  $\rho = \top$  since  $\rho \leq \tau$ , and we cannot have  $\rho = \perp$  since  $\sigma \leq \rho$ , therefore  $\rho = \rho_1 \rightarrow \rho_2$  for some  $\rho_1, \rho_2$ . Then we must also have  $\sigma = \sigma_1 \rightarrow \sigma_2$  since  $\sigma \leq \rho$  and  $\tau = \tau_1 \rightarrow \tau_2$  since  $\rho \leq \tau$ . Because  $\sigma \leq \rho \leq \tau$ , we must have  $\tau_1 \leq \rho_1 \leq \sigma_1$  and  $\sigma_2 \leq \rho_2 \leq \tau_2$ . By the coinduction hypothesis,  $\tau_1 \leq \sigma_1$  and  $\sigma_2 \leq \tau_2$ , therefore  $\sigma \leq \tau$ .  $\square$

## 3.3 Closure Conversion

Here is a more involved example from [7]. Consider the  $\lambda$ -calculus with variables  $\text{Var}$  and atomic constants  $\text{Const}$ . Let  $\text{FV}(e)$  denote the set of free variables of  $e$ . Let  $\lambda\text{-Abs}$  denote the set of  $\lambda$ -abstractions  $\lambda x.e$ . A *capsule* is a pair  $\langle e, \sigma \rangle$ , where  $e$  is a  $\lambda$ -term and  $\sigma : \text{Var} \rightarrow \text{Const} + \lambda\text{-Abs}$  is a partial function with finite domain  $\text{dom } \sigma$ , such that

- (i)  $\text{FV}(e) \subseteq \text{dom } \sigma$
- (ii) if  $x \in \text{dom } \sigma$ , then  $\text{FV}(\sigma(x)) \subseteq \text{dom } \sigma$ .

The component  $\sigma$  is called a *capsule environment*. The set of capsule environments is denoted  $\text{CapEnv}$ .

Capsules are a coalgebraic representation the state of a computation in functional and imperative languages. They minimally extend the  $\lambda$ -calculus to allow variables and assignment. Capsules are essentially coalgebras, and in [7] informal coinductive reasoning was used extensively.



One result from [7] was that closure conversion for capsules is correct. *Closures* are the traditional representation of state, originally formulated to achieve static scoping. They can be defined coinductively with the recursive type definition

$$\begin{aligned} \text{Val} &= \text{Const} + \text{Cl} && \text{values} \\ \text{Cl} &= \lambda\text{-Abs} \times \text{ClEnv} && \text{closures} \\ \text{ClEnv} &= \text{Var} \rightarrow \text{Val} && \text{closure environments} \end{aligned}$$

Thus a closure is a pair  $\{\lambda x.e, \sigma\}$ , where  $\lambda x.e$  is a  $\lambda$ -abstraction and  $\sigma$  is a closure environment. We use boldface for closure environments  $\sigma : \text{ClEnv}$  to distinguish them from the simpler capsule environments. Closures  $\{\lambda x.e, \sigma\}$  must satisfy the additional requirements that  $\text{FV}(\lambda x.e) \subseteq \text{dom } \sigma$  and  $\text{dom } \sigma$  is finite.

The closure-converted form of a capsule  $\langle e, \sigma \rangle$  is  $\langle e, \bar{\sigma} \rangle$ , where

$$\bar{\sigma}(y) = \begin{cases} \{\sigma(y), \bar{\sigma}\}, & \text{if } \sigma(y) : \lambda\text{-Abs}, \\ \sigma(y), & \text{if } \sigma(y) : \text{Const}. \end{cases}$$

This is a coinductive definition. Although it appears circular, it actually defines  $\bar{\sigma}$  uniquely for the same reason that  $+$  was defined uniquely in §3.1. In pseudo-ML, the definition might look like

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let rec  $\bar{\sigma} = \lambda y. \text{match } \sigma(y) \text{ with}$ 
  |  $\text{Const}(c) \rightarrow \text{Const}(c)$ 
  |  $\lambda\text{-Abs}(\lambda x.e) \rightarrow \text{Cl}(\{\lambda x.e, \bar{\sigma}\})$ 
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To state the relationship between capsules and closures, we define a binary relation  $\sqsubseteq$  on capsule environments, closure environments, and values. For capsule environments, define  $\sigma \sqsubseteq \tau$  if  $\text{dom } \sigma \subseteq \text{dom } \tau$  and for all  $y \in \text{dom } \sigma$ ,  $\sigma(y) = \tau(y)$ . The definition for values and closure environments is by mutual coinduction:  $\sqsubseteq$  is defined to be the largest relation such that

- A. on closure environments, if  $\sigma \sqsubseteq \tau$  then
  - (i)  $\text{dom } \sigma \subseteq \text{dom } \tau$ , and
  - (ii) for all  $y \in \text{dom } \sigma$ ,  $\sigma(y) \sqsubseteq \tau(y)$ ; and
- B. on values, if  $u \sqsubseteq v$  then
  - (i)  $u$  and  $v$  are constants and  $u = v$ ; or
  - (ii)  $u = \{\lambda x.e, \sigma\}$ ,  $v = \{\lambda x.e, \tau\}$ , and  $\sigma \sqsubseteq \tau$ .

Formally,  $\sqsubseteq$  for closures consists of two relations defined by mutual coinduction, one on closure environments and one on values. More precisely, the relation  $\sqsubseteq$  is defined to be the largest relation  $R$  on  $(\text{ClEnv} \times \text{ClEnv}) + (\text{Val} \times$

Val) such that  $R \subseteq T(R)$  (symbolically,  $\sqsubseteq = \nu X.T(X)$ ), where  $T$  is the monotone map

$$T: (\text{CIEnv} \times \text{CIEnv}) + (\text{Val} \times \text{Val}) \rightarrow (\text{CIEnv} \times \text{CIEnv}) + (\text{Val} \times \text{Val})$$

defined as follows:

- A. for closure environments,  $(\sigma, \tau) \in T(R)$  iff
  - (i)  $\text{dom } \sigma \subseteq \text{dom } \tau$ , and
  - (ii) for all  $y \in \text{dom } \sigma$ ,  $(\sigma(y), \tau(y)) \in R$ ; and
- B. for values,  $(u, v) \in T(R)$  iff either
  - (i)  $u$  and  $v$  are constants and  $u = v$ ; or
  - (ii)  $u = \{\lambda x.e, \sigma\}$ ,  $v = \{\lambda x.e, \tau\}$ , and  $(\sigma, \tau) \in R$ .

**Theorem 3.5** *The relation  $\sqsubseteq$  is transitive.*

*Proof.* Suppose  $\sigma \sqsubseteq \rho \sqsubseteq \tau$ . By A(i),  $\text{dom } \sigma \subseteq \text{dom } \rho \subseteq \text{dom } \tau$ , so  $\text{dom } \sigma \subseteq \text{dom } \tau$ , and A(i) holds for the pair  $\sigma, \tau$ . Moreover, for all  $y \in \text{dom } \sigma$ , by A(ii),  $\sigma(y) \sqsubseteq \rho(y) \sqsubseteq \tau(y)$ , therefore  $\sigma(y) \sqsubseteq \tau(y)$  by the coinduction hypothesis on values. Thus A(ii) holds, and  $\sigma \sqsubseteq \tau$ .

For values, suppose  $u \sqsubseteq w \sqsubseteq v$ . If  $u$  is a constant  $c$ , then  $w = c$  and  $v = c$ , hence B(i) holds for the pair  $u, v$ . If  $u = \{\lambda x.e, \sigma\}$ , then by B(ii),  $w = \{\lambda x.e, \rho\}$ ,  $v = \{\lambda x.e, \tau\}$ , and  $\sigma \sqsubseteq \rho \sqsubseteq \tau$ . By the coinduction hypothesis on closure environments,  $\sigma \sqsubseteq \tau$ , thus  $u \sqsubseteq v$ .  $\square$

**Theorem 3.6** *Closure conversion is monotone with respect to  $\sqsubseteq$ . That is, if  $\sigma \sqsubseteq \tau$ , then  $\bar{\sigma} \sqsubseteq \bar{\tau}$ .*

*Proof.* Let  $\sigma$  and  $\tau$  be capsule environments and suppose that  $\sigma \sqsubseteq \tau$ . Then  $\text{dom } \sigma \subseteq \text{dom } \tau$  and  $\sigma(y) = \tau(y)$  for all  $y \in \text{dom } \sigma$ . Note that  $\text{dom } \bar{\sigma} = \text{dom } \sigma \subseteq \text{dom } \tau = \text{dom } \bar{\tau}$ , which gives A(i) for  $\bar{\sigma}$  and  $\bar{\tau}$  immediately.

For any  $y \in \text{dom } \bar{\sigma}$ , if  $\sigma(y)$  is a constant  $c$ , then  $\tau(y) = c$  because  $\sigma \sqsubseteq \tau$ , and  $\bar{\sigma}(y) = \sigma(y) = \tau(y) = \bar{\tau}(y)$ , thus A(ii) holds of  $\bar{\sigma}$  and  $\bar{\tau}$ . If  $\sigma(y)$  is a  $\lambda$ -abstraction, then so is  $\tau(y)$  and they are equal, thus  $\bar{\sigma}(y) = \{\sigma(y), \bar{\sigma}\} \sqsubseteq \{\tau(y), \bar{\tau}\} = \bar{\tau}(y)$ , using the coinduction hypothesis B(ii). In both cases, A(ii) holds of  $\bar{\sigma}$  and  $\bar{\tau}$ .  $\square$

### 3.4 Bisimilarity

The traditional coinduction principle states that if two elements of a final coalgebra are bisimilar, then they are equal. More generally, bisimilarity between two elements of two coalgebras implies that their unique images in the final

coalgebra are the same. These traditional proofs can also be handled by our method.

For example, in the case of  $A$ -streams,  $R$  is a bisimulation if for any  $\sigma, \tau$ ,

$$(\sigma, \tau) \in R \Rightarrow \text{hd}(\sigma) = \text{hd}(\tau) \text{ and } (\text{tl}(\sigma), \text{tl}(\tau)) \in R.$$

The relation of bisimilarity  $\sim$  is the maximal bisimulation. This is the greatest postfixpoint of the monotone operator

$$T(R) = \{(\sigma, \tau) \mid \text{hd}(\sigma) = \text{hd}(\tau) \text{ and } (\text{tl}(\sigma), \text{tl}(\tau)) \in R\},$$

or in other words, the greatest relation  $\sim$  such that  $\sim \subseteq T(\sim)$ . The greatest postfixpoint is also the greatest fixpoint, therefore  $\sim = T(\sim)$ .

We can now prove coinductively that  $\sim$  is an equivalence relation. Let us illustrate by proving that  $\sim$  on streams is symmetric.

**Theorem 3.7**  $\sim$  on  $A$ -streams is symmetric relation. That is,  $\sigma \sim \tau$  implies  $\tau \sim \sigma$ .

*Proof.* Assume  $\sigma \sim \tau$ . Then  $\text{hd}(\sigma) = \text{hd}(\tau)$  and  $\text{tl}(\sigma) \sim \text{tl}(\tau)$ . By the symmetry of equality on  $A$ ,  $\text{hd}(\tau) = \text{hd}(\sigma)$ . By the coinduction hypothesis on the tails,  $\text{tl}(\tau) \sim \text{tl}(\sigma)$ . As  $\sim$  is maximal,  $\tau \sim \sigma$ .  $\square$

We can also use the principle to reason about properties of stream operations. For example, consider the two inverse stream operations

$$\begin{aligned} \text{split}(\sigma_0\sigma_1\sigma_2\cdots) &= (\sigma_0\sigma_2\dots, \sigma_1\sigma_3\cdots) \\ \text{merge}(\sigma_0\sigma_1\cdots, \tau_0\tau_1\cdots) &= \sigma_0\tau_0\sigma_1\tau_1\cdots, \end{aligned}$$

characterized coinductively by the equations

$$\begin{aligned} \text{merge}(a :: \sigma, \tau) &= a :: \text{merge}(\tau, \sigma) \\ \text{split}(a :: \sigma) &= \text{let } (\rho, \tau) = \text{split}(\sigma) \text{ in } (a :: \tau, \rho) \end{aligned}$$

or, expressed completely in terms of destructors,

$$\begin{aligned} \text{hd}(\text{merge}(\sigma, \tau)) &= \text{hd}(\sigma) & \text{hd}(\text{split}(\sigma)_1) &= \text{hd}(\sigma) \\ \text{tl}(\text{merge}(\sigma, \tau)) &= \text{merge}(\tau, \text{tl}(\sigma)) & \text{tl}(\text{split}(\sigma)_1) &= \text{split}(\text{tl}(\sigma))_2 \\ & & \text{split}(\sigma)_2 &= \text{split}(\text{tl}(\sigma))_1. \end{aligned} \quad (3.5)$$

Let us argue that merge is a left inverse of split.

**Theorem 3.8** For all streams  $\sigma$ ,  $\text{merge}(\text{split}(\sigma)) = \sigma$ .

*Proof.* We argue in terms of the characterization (3.5).

$$\begin{aligned}
\text{hd}(\text{merge}(\text{split}(\sigma))) &= \text{hd}(\text{merge}(\text{split}(\sigma)_1, \text{split}(\sigma)_2)) \\
&= \text{hd}(\text{split}(\sigma)_1) \\
&= \text{hd}(\sigma) \\
\text{tl}(\text{merge}(\text{split}(\sigma))) &= \text{tl}(\text{merge}(\text{split}(\sigma)_1, \text{split}(\sigma)_2)) \\
&= \text{merge}(\text{split}(\sigma)_2, \text{tl}(\text{split}(\sigma)_1)) \\
&= \text{merge}(\text{split}(\text{tl}(\sigma))_1, \text{split}(\text{tl}(\sigma))_2) \\
&= \text{merge}(\text{split}(\text{tl}(\sigma))) \\
&\sim \text{tl}(\sigma),
\end{aligned}$$

the last step by the coinduction hypothesis. As  $\sim$  is maximal, we can conclude that  $\text{merge}(\text{split}(\sigma)) \sim \sigma$ . Since equality and bisimilarity coincide on the final coalgebra,  $\text{merge}(\text{split}(\sigma)) = \sigma$ .  $\square$

Why did we not argue in the last step that  $\text{merge}(\text{split}(\text{tl}(\sigma))) = \text{tl}(\sigma)$  by the coinduction hypothesis, then conclude that  $\text{merge}(\text{split}(\sigma)) = \sigma$  because the heads and tails were equal? We might have done so, but we wanted to emphasize that it is bisimilarity  $\sim$ , not equality  $=$ , that is the maximal fixpoint of the relevant monotone map

$$T(X) = \{(\sigma, \tau) \mid \text{hd}(\sigma) = \text{hd}(\tau) \text{ and } (\text{tl}(\sigma), \text{tl}(\tau)) \in X\}.$$

We may not use the technique with just any property, only with those defined as maximal fixpoints.

We could conclude equality in the last sentence because streams are the final coalgebra, for which bisimilarity and equality coincide. But except for the last sentence, the argument works for any coalgebra for this signature. Consider coalgebras  $(X, \text{obs}, \text{cont})$  with observations  $\text{obs} : X \rightarrow A$  and continuations  $\text{cont} : X \rightarrow X$ . The equations (3.5) can be interpreted as implicit coinductive descriptions of maps  $\text{merge} : C \times C \rightarrow C$  and  $\text{split} : C \rightarrow C \times C$ :

$$\begin{aligned}
\text{obs}(\text{merge}(x, y)) &= \text{obs}(x) & \text{obs}(\text{split}_1(x)) &= \text{obs}(x) \\
\text{cont}(\text{merge}(x, y)) &= \text{merge}(y, \text{cont}(x)) & \text{cont}(\text{split}_1(x)) &= \text{split}_2(\text{cont}(x)) \\
& & \text{split}_2(x) &= \text{split}_1(\text{cont}(x)).
\end{aligned}$$

Note that these equations do not define  $\text{merge}$  and  $\text{split}$  uniquely, because they do not specify what  $\text{merge}(x, y)$  and  $\text{split}(x)$  are, but only describe their observable behavior. Nevertheless, whatever they are, they are inverses up to bisimulation:

**Theorem 3.9** *For all  $x$ ,  $\text{merge}(\text{split}(x)) \sim x$ .*

The proof is the same as that of Theorem 3.8, *mutatis mutandis*.

## 4 A Coinductive Proof Principle

The proofs of §3, magical as they may seem, involve no magic—only a little sleight of hand! The rule we are using is best explained in the language of dynamic logic (DL) and the modal  $\mu$ -calculus (see [3]). Our examples typically involve

- coalgebras  $K_1, K_2$  viewed as Kripke models with binary relations  $a$  and  $b$ , respectively, encoding coalgebraic destructors, and
- a kind of simulation relation  $\pi : K_1 \times K_2$  between them, often a function  $\pi : K_1 \rightarrow K_2$ .

The relations  $a$  and  $b$  induce modalities  $[a], \langle a \rangle$  on  $K_1$  and  $[b], \langle b \rangle$  on  $K_2$ . To have a common domain to work in, we form the coproduct  $K = K_1 + K_2$  whose elements are the disjoint union of  $K_1$  and  $K_2$  with relations  $a$  and  $b$  inherited from  $K_1$  and  $K_2$ .

We are typically trying to establish that a property of the form  $Q \rightarrow [\pi]R$  holds universally in  $K$ , where  $R$  is a property on  $K_2$  defined as a greatest fix-point of the form  $R = \nu X.G \wedge [b]X = [b^*]G$  and  $Q$  is a precondition defined on  $K_1$ . The property  $Q \rightarrow [\pi]R$  says that any state in  $K_1$  satisfying  $Q$  must map under  $\pi$  to a state or states in  $K_2$  satisfying  $R$ . The property  $G$  in the definition of  $R$  is typically a condition that can be checked locally on states of  $K_2$ , whereas the part of the definition involving  $[b]$  encodes a recursive check of  $R$  on successor states.

For example, in the application of §3.1 involving the transitivity of  $\leq_{\text{lex}}$  on  $A$ -streams, the statement we are trying to prove is:

For all  $A$ -streams  $\sigma, \rho, \tau$ , if  $\sigma \leq_{\text{lex}} \rho \leq_{\text{lex}} \tau$ , then  $\sigma \leq_{\text{lex}} \tau$ .

Here  $K_1 = A^\omega \times A^\omega \times A^\omega$  and  $K_2 = A^\omega \times A^\omega$ , along with relations

$$\begin{aligned} (\sigma, \rho, \tau) &\xrightarrow{a} (\text{tl}(\sigma), \text{tl}(\rho), \text{tl}(\tau)), \text{ if } \text{hd}(\sigma) = \text{hd}(\rho) = \text{hd}(\tau) \\ (\sigma, \tau) &\xrightarrow{b} (\text{tl}(\sigma), \text{tl}(\tau)), \text{ if } \text{hd}(\sigma) = \text{hd}(\tau) \\ (\sigma, \rho, \tau) &\xrightarrow{\pi} (\sigma, \tau). \end{aligned}$$

In this case the relation  $\pi$  is a function  $\pi : K_1 \rightarrow K_2$ , the projection of a triple onto its first and third components.

The property  $Q$  is true of a triple  $(\sigma, \rho, \tau)$  if  $\sigma \leq_{\text{lex}} \rho \leq_{\text{lex}} \tau$ , and the property  $R$  is true of a pair  $(\sigma, \tau)$  if  $\sigma \leq_{\text{lex}} \tau$ . Transitivity states that  $Q \rightarrow [\pi]R$  is universally valid in  $K$ . The definition of  $R$  is  $R = \nu X.G \wedge [b]X = [b^*]G$ , where

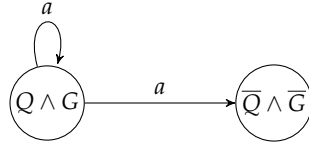
$$G = \{(\sigma, \tau) \mid \text{hd}(\sigma) \leq \text{hd}(\tau)\}.$$

## 4.1 A Proof Rule

It may seem that the informal rule we are using is

$$\frac{Q \rightarrow [\pi]G \quad [a](Q \rightarrow [\pi]R) \rightarrow (Q \rightarrow [\pi]R)}{Q \rightarrow [\pi]R}. \quad (4.6)$$

However, this rule is unsound in general (this is the sleight of hand mentioned above). Here is a counterexample, in which  $a = b$  and  $\pi$  is the identity:



But a careful look at the proof reveals that we did not use any properties of  $R$  except  $G \wedge [b]R \rightarrow R$  at the very last moment. Up to that point, the induction step actually established that

$$[a](Q \rightarrow [\pi]X) \rightarrow (Q \rightarrow [\pi b]X) \quad (4.7)$$

without any knowledge of  $X$ . Thus we are actually using the rule

$$\frac{Q \rightarrow [\pi]G \quad [a](Q \rightarrow [\pi]X) \rightarrow (Q \rightarrow [\pi b]X)}{Q \rightarrow [\pi]R}, \quad (4.8)$$

where  $X$  is a fresh atomic symbol. We prove below (Theorem 4.1) that this rule is sound.

Rules similar to this appear in different forms in the literature [2, 6, 22]. In most cases, the rules are Gentzen-style with structural restrictions such as *progress* (aka *guardedness* or *contraction* [2]) and *opacity* (aka *frozenness* [22]). In our treatment, progress takes the form of the modalities  $[a]$ ,  $[b]$ , and opacity is captured in the use of the atomic symbol  $X$ .

We have mentioned that we engaged in a little sleight-of-hand. This has to do with the use of  $R$  instead of  $X$  in the last step, which makes it seem as if we are using the unsound rule (4.6). To be completely honest, in the proof of Theorem 3.2 we should replace the sentence,

By the coinduction hypothesis,  $\text{tl}(\sigma) \leq_{\text{lex}} \text{tl}(\tau)$ .

with

By the coinduction hypothesis,  $(\text{tl}(\sigma), \text{tl}(\tau)) \in X$ , thus  $(\sigma, \tau) \in [b]X$  and  $(\sigma, \rho, \tau) \in [\pi b]X$ .

## 4.2 Soundness

**Theorem 4.1** *The rule (4.8) is sound.*

We give two proofs of this theorem.

*Proof 1.* For any  $P$ , if  $K \models Q \rightarrow [\pi]P$ , then  $K \models [a](Q \rightarrow [\pi]P)$  by modal generalization. Substituting  $P$  for  $X$  in the second premise of (4.8), we have  $K \models Q \rightarrow [\pi b]P$ . Thus for any  $P$ ,

$$K \models Q \rightarrow [\pi]P \Rightarrow K \models Q \rightarrow [\pi b]P.$$

Applying this construction inductively, we have that for all  $P$  and all  $n \geq 0$ ,

$$K \models Q \rightarrow [\pi]P \Rightarrow K \models Q \rightarrow [\pi b^n]P,$$

therefore

$$K \models Q \rightarrow [\pi]P \Rightarrow K \models Q \rightarrow [\pi b^*]P.$$

In particular, for  $P = G$ , using the first premise of (4.8) and the definition  $R = [b^*]G$ , we conclude that  $K \models Q \rightarrow [\pi]R$ .  $\square$

*Proof 2.* From DL, we have the Galois connection

$$\models X \rightarrow [c]Y \Leftrightarrow \models \langle c^- \rangle X \rightarrow Y, \quad (4.9)$$

where  $c^- = \{(s, t) \mid (t, s) \in c\}$ . Specializing the second premise of (4.8) at  $X = \langle \pi^- \rangle Q$ , we have

$$K \models [a](Q \rightarrow [\pi]\langle \pi^- \rangle Q) \rightarrow (Q \rightarrow [\pi b]\langle \pi^- \rangle Q).$$

But the left-hand side is a tautology of DL, therefore by modus ponens this reduces to  $K \models Q \rightarrow [\pi b]\langle \pi^- \rangle Q$ . Again by (4.9) we have

$$K \models \langle \pi^- \rangle Q \rightarrow [b]\langle \pi^- \rangle Q.$$

Similarly, applying (4.9) to the first premise of (4.8), we have  $K \models \langle \pi^- \rangle Q \rightarrow G$ . Combining these two facts,

$$K \models \langle \pi^- \rangle Q \rightarrow G \wedge [b]\langle \pi^- \rangle Q,$$

therefore  $K \models \langle \pi^- \rangle Q \rightarrow R$ , since  $R = \nu X.G \wedge [b]X$ . Applying (4.9) one more time, we obtain  $K \models Q \rightarrow [\pi]R$ , the conclusion of (4.8).  $\square$

### 4.3 A More General Version

The rule (4.8) only applies to monotone transformations of the form  $T(X) = G \wedge [b]X$ , for which  $R = \nu X.T(X) = [b^*]G$ . This is all we need for the examples in this paper. However, one can generalize the rule to arbitrary monotone  $T$  at the expense of some added complication in the proof system. The rule is

$$\frac{\Gamma, Q \rightarrow [\pi]X \vdash Q \rightarrow [\pi]T(X)}{\Gamma \vdash Q \rightarrow [\pi]R}, \quad (4.10)$$

for  $X$  a fresh atomic symbol not occurring in  $\Gamma$ ,  $Q$ , or  $\pi$ , where  $R = \nu X.T(X)$ . In other words, if it is possible to prove  $Q \rightarrow [\pi]T(X)$  from the assumptions  $\Gamma$  and  $Q \rightarrow [\pi]X$ , where  $X$  is an atomic symbol not occurring elsewhere, then it is safe to conclude  $Q \rightarrow [\pi]R$ . Soundness would say that for any Kripke model  $K$  satisfying  $\Gamma$ , if  $K \models Q \rightarrow [\pi]T(X)$  whenever  $K \models Q \rightarrow [\pi]X$ , then  $K \models Q \rightarrow [\pi]R$ . This rule now looks more like the rules of [2, 6, 22].

**Theorem 4.2** *The rule (4.10) is sound.*

*Proof.* By induction on the lengths of proofs. Suppose it is possible to prove  $Q \rightarrow [\pi]T(X)$  from the assumptions  $\Gamma$  and  $Q \rightarrow [\pi]X$ , where  $X$  is an atomic symbol not occurring in  $\Gamma$ ,  $Q$ , or  $\pi$ . By the induction hypothesis, that proof is sound. Thus in any Kripke model  $K$  satisfying  $\Gamma$ , for any interpretation of  $X$ ,

$$K \models Q \rightarrow [\pi]X \Rightarrow K \models Q \rightarrow [\pi]T(X).$$

In particular, for  $X = \langle \pi^- \rangle Q$ , we have

$$K \models Q \rightarrow [\pi] \langle \pi^- \rangle Q \Rightarrow K \models Q \rightarrow [\pi]T(\langle \pi^- \rangle Q).$$

The left-hand side is a tautology of DL, so we are left with the right-hand side, which reduces by (4.9) to  $K \models \langle \pi^- \rangle Q \rightarrow T(\langle \pi^- \rangle Q)$ . As  $R = \nu X.T(X)$  is the greatest postfixpoint of  $T$ , we have  $K \models \langle \pi^- \rangle Q \rightarrow R$ . The conclusion  $K \models Q \rightarrow [\pi]R$  follows from this and (4.9).  $\square$

### 4.4 Examples

We now describe how the other examples of §3 fit into this framework.

#### 4.4.1 Recursive Types

In the example of §3.2, the statement we are trying to prove is

$$\text{For all types } \sigma, \rho, \tau, \text{ if } \sigma \leq \rho \leq \tau, \text{ then } \sigma \leq \tau.$$



Here  $K_1 = C \times C \times C$  and  $K_2 = C \times C$ , where  $C$  is the set of recursive types, along with relations

$$\begin{array}{ccc}
 (\sigma_1 \rightarrow \sigma_2, \rho_1 \rightarrow \rho_2, \tau_1 \rightarrow \tau_2) & & (\sigma_1 \rightarrow \sigma_2, \tau_1 \rightarrow \tau_2) \\
 \swarrow a & & \swarrow b \\
 (\tau_1, \rho_1, \sigma_1) & & (\tau_1, \sigma_1) \\
 \searrow a & & \searrow b \\
 (\sigma_2, \rho_2, \tau_2) & & (\sigma_2, \tau_2)
 \end{array}$$

$$(\sigma, \rho, \tau) \xrightarrow{\pi} (\sigma, \tau)$$

The relation  $\pi$  is a function  $\pi : K_1 \rightarrow K_2$ , the projection of a triple onto its first and third components. Note the contravariance of the left-hand  $a$ - and  $b$ -successors.

The property  $Q$  is true of a triple  $(\sigma, \rho, \tau)$  if  $\sigma \leq \rho \leq \tau$ , and the property  $R$  is true of a pair  $(\sigma, \tau)$  if  $\sigma \leq \tau$ . Transitivity states that  $Q \rightarrow [\pi]R$ . The definition of  $R$  is  $R = \nu X.G \wedge [b]X = [b^*]G$ , where

$$G = \{(\sigma, \tau) \mid [b]\text{false} \rightarrow (\sigma = \perp \vee \tau = \top)\},$$

that is,  $G$  holds of a pair  $(\sigma, \tau)$  with no  $b$ -successors in  $K_2$  if either  $\sigma = \perp$  or  $\tau = \top$ , thus  $\sigma \leq \tau$  by local considerations.

Note that there can be an infinite  $b$ -path of pairs  $(\sigma, \tau)$  such that  $\sigma \not\leq \tau$ . For example, if  $\sigma = \perp \rightarrow \sigma$  and  $\tau = \top \rightarrow \tau$ , then  $\sigma \not\leq \tau$  and  $(\sigma, \tau) \xrightarrow{b} (\sigma, \tau)$ .

The property  $\langle \pi^- \rangle Q$  in the second proof of Theorem 4.1 is true of the pair  $(\sigma, \tau)$  iff  $\exists \rho \sigma \leq \rho \leq \tau$ . The main part of the argument of Theorem 3.4 essentially shows that  $\langle \pi^- \rangle Q \rightarrow [b]\langle \pi^- \rangle Q$  and that  $\langle \pi^- \rangle Q \rightarrow G$ , thereby establishing that  $\langle \pi^- \rangle Q$  is a postfixpoint of  $T(X) = G \wedge [b]X$ .

#### 4.4.2 Closure Conversion

In the example of §3.3 involving the monotonicity of closure conversion, recall that the closure-converted form of a capsule  $\langle e, \sigma \rangle$  is  $\langle e, \bar{\sigma} \rangle$ , where  $\bar{\sigma}$  is defined as

$$\bar{\sigma}(y) = \begin{cases} \{\sigma(y), \bar{\sigma}\}, & \text{if } \sigma(y) : \lambda\text{-Abs}, \\ \sigma(y), & \text{if } \sigma(y) : \text{Const}. \end{cases}$$

Here we can take

$$K_1 = \text{CapEnv} \times \text{CapEnv} \qquad K_2 = \text{CIEEnv} \times \text{CIEEnv},$$

where  $\text{CapEnv}$  and  $\text{CIEEnv}$  are the sets of capsule environments and closure environments, respectively, and

$$Q = \{(\sigma, \tau) \mid \sigma \sqsubseteq \tau\} \qquad R = \{(\sigma, \tau) \mid \sigma \sqsubseteq \tau\}$$

The relation  $\sqsubseteq$  on capsule environments can be defined without coinduction:  $\sigma \sqsubseteq \tau$  if  $\text{dom } \sigma \subseteq \text{dom } \tau$  and for all  $y \in \text{dom } \sigma$ ,  $\sigma(y) = \tau(y)$ . The definition for closure environments is by coinduction. In §3.3, it was defined by mutual coinduction on closure environments and values, but we can consolidate this into a definition just on closure environments:  $\sqsubseteq$  is the largest binary relation on closure environments such that if  $\sigma \sqsubseteq \tau$ , then  $\text{dom } \sigma \subseteq \text{dom } \tau$ , and for all  $y \in \text{dom } \sigma$ , either

- $\sigma(y)$  and  $\tau(y)$  are constants and  $\sigma(y) = \tau(y)$ ; or
- $\sigma(y) = \{\lambda x.e, \rho\}$ ,  $\tau(y) = \{\lambda x.e, \pi\}$ , and  $\rho \sqsubseteq \pi$ .

The relation  $R$  is defined as the greatest fixpoint  $\nu X.G \wedge [b]X = [b^*]G$ , where  $G$  is true of a pair  $(\sigma, \tau)$  if  $\text{dom } \sigma \subseteq \text{dom } \tau$  and for all for all  $y \in \text{dom } \sigma$ , either

- $\sigma(y)$  and  $\tau(y)$  are constants and  $\sigma(y) = \tau(y)$ ; or
- $\sigma(y) = \{\lambda x.e, \rho\}$  and  $\tau(y) = \{\lambda x.e, \pi\}$  for some  $\lambda x.e, \rho$ , and  $\pi$ ,

and the relation  $b$  on  $K_2$  is

$$(\sigma, \tau) \xrightarrow{b} (\rho, \pi)$$

whenever  $\sigma(y) = \{d, \rho\}$  and  $\tau(y) = \{e, \pi\}$  for some  $d, e$ , and  $y$ . The relation  $a$  on  $K_1$  is simply  $(\sigma, \tau) \xrightarrow{a} (\sigma, \tau)$ .

The monotonicity theorem says

$$\forall \sigma \forall \tau \sigma \sqsubseteq \tau \rightarrow \bar{\sigma} \sqsubseteq \bar{\tau},$$

which is just  $Q \rightarrow [\pi]R$ , where  $\pi$  is the closure conversion function  $\sigma \mapsto \bar{\sigma}$ .

#### 4.4.3 Bisimilarity

In §3.4, we proved that bisimilarity is symmetric on streams and that merge is a left inverse of split.

In the first example, the statement we are trying to prove is

$$\text{For all } A\text{-streams } \sigma, \tau, \text{ if } \sigma \sim \tau, \text{ then } \tau \sim \sigma.$$

Here we take  $K_1 = K_2 = A^\omega \times A^\omega$  with relations

$$(\sigma, \tau) \xrightarrow{a,b} (\text{tl}(\sigma), \text{tl}(\tau)) \qquad (\sigma, \tau) \xrightarrow{\pi} (\tau, \sigma).$$

The properties  $Q$  and  $R$  are both  $\sim$ . The theorem states that  $Q \rightarrow [\pi]R$ . The definition of  $R$  is  $R = \nu X.G \wedge [b]X = [b^*]G$ , where

$$G = \{(\sigma, \tau) \mid \text{hd}(\sigma) = \text{hd}(\tau)\}.$$

In the second example, the statement we are trying to prove is

For all  $A$ -streams  $\sigma$ ,  $\text{merge}(\text{split}(\sigma)) = \sigma$ .

Here we take  $K_1 = A^\omega$  and  $K_2 = A^\omega \times A^\omega$  with relations

$$\sigma \xrightarrow{a} \text{tl}(\sigma) \quad (\sigma, \tau) \xrightarrow{b} (\text{tl}(\sigma), \text{tl}(\tau)) \quad \sigma \xrightarrow{\pi} (\text{merge}(\text{split}(\sigma)), \sigma).$$

The property  $R$  is still  $\sim$  as above, but here  $Q = \text{true}$ . In this case the theorem  $Q \rightarrow [\pi]R$  reduces to  $[\pi]R$ . It is interesting to note that the property  $\langle \pi^- \rangle Q$  in the second proof of Theorem 4.1 here reduces to  $\langle \pi^- \rangle \text{true}$  and holds of a pair  $(\sigma, \tau)$  iff  $\sigma = \text{merge}(\text{split}(\tau))$ .

## 4.5 Discussion

There are two sufficient conditions for the premise (4.7) of our proof rule that hold in many applications. These conditions can be expressed in the language of Kleene algebra with tests (KAT) [9]. They are

$$Q\pi b \leq Qa\pi \quad Qa \leq aQ. \quad (4.11)$$

The condition on the left says that the relation  $\pi$  is a kind of simulation: under the enabling condition  $Q$ , the action  $a$  on the left-hand side of  $\pi$  simulates the action  $b$  on the right-hand side. It serves the same purpose as the DL formula  $Q \rightarrow [a\pi]X \rightarrow [\pi b]X$  for atomic  $X$ , but is slightly stronger.

**Lemma 4.3** *In any Kripke model  $K$ , if  $Q\pi b \leq Qa\pi$ , then for any  $X$ , the DL formula  $Q \rightarrow [a\pi]X \rightarrow [\pi b]X$  holds universally in  $K$ .*

*Proof.* Suppose  $Q\pi b \leq Qa\pi$  in  $K$ . Then for any  $X$ ,  $Q\pi b\bar{X} \leq a\pi\bar{X}$ , where the overbar denotes Boolean negation. This implies the DL formula  $Q \wedge \langle \pi b \rangle \bar{X} \rightarrow \langle a\pi \rangle \bar{X}$ , which is equivalent to  $Q \rightarrow [a\pi]X \rightarrow [\pi b]X$ .  $\square$

The condition on the right of (4.11) says that the enabling condition  $Q$  is preserved by  $a$  on the left-hand side. It is equivalent to the KAT equations  $Qa = QaQ$  and  $Qa\bar{Q} = 0$ , to the DL formula  $Q \rightarrow [a]Q$ , and to the Hoare partial correctness assertion  $\{Q\} a \{Q\}$ .

**Theorem 4.4** *If  $Q\pi b \leq Qa\pi$ , then the formula*

$$(Q \rightarrow [\pi]G) \rightarrow (Q \rightarrow [a]Q) \rightarrow [a](Q \rightarrow [\pi]R) \rightarrow (Q \rightarrow [\pi]R)$$

*is universally valid.*

*Proof.* We show that any state satisfying  $Q \rightarrow [\pi]G$ ,  $Q \rightarrow [a]Q$ ,  $[a](Q \rightarrow [\pi]R)$ , and  $Q$  also satisfies  $[\pi]R$ . From  $Q$  and  $Q \rightarrow [a]Q$  we have  $[a]Q$ . From  $[a]Q$  and  $[a](Q \rightarrow [\pi]R)$  we have  $[a](Q \wedge (Q \rightarrow [\pi]R))$ , whence  $[a\pi]R$ . From  $Q$  and  $[a\pi]R$ , by Lemma 4.3 we have  $[\pi b]R$ . From  $Q$  and  $Q \rightarrow [\pi]G$  we have  $[\pi]G$ . From  $[\pi b]R$  and  $[\pi]G$  we have  $[\pi](G \wedge [b]R)$ , and since  $G \wedge [b]R = R$  we have  $[\pi]R$ .  $\square$

It follows from Theorem 4.4 that if  $Q\pi b \leq Qa\pi$ , then the proof rule

$$\frac{Q \rightarrow [\pi]G \quad Q \rightarrow [a]Q \quad [a](Q \rightarrow [\pi]R)}{Q \rightarrow [\pi]R}$$

is sound, and this rule is similar to our unsound rule (4.6). However, in this case a stronger result holds.

**Lemma 4.5** *The following is a theorem of KAT:*

$$Q\pi b \leq Qa\pi \wedge Qa \leq aQ \rightarrow Q\pi b^* \leq Qa^*\pi.$$

*Proof.* The premise  $Qa \leq aQ$  is equivalent to  $Qa = QaQ$ . Using this,

$$Q + (Qa)^*Qa = Q + (Qa)^*QaQ = (Qa)^*Q,$$

therefore by a star rule of Kleene algebra,  $Qa^* \leq (Qa)^*Q$ . Also,  $(Qa)^*Q = Q(aQ)^* \leq Qa^*$ , therefore

$$Qa^* = (Qa)^*Q = Q(aQ)^*.$$

Using this and the first premise, we have

$$Qa^*\pi b = (Qa)^*Q\pi b = (Qa)^*Qa\pi = Qa^*a\pi,$$

therefore

$$Q\pi + Qa^*\pi b = Q\pi + Qa^*a\pi = Qa^*\pi.$$

Again by a star rule,  $Q\pi b^* \leq Qa^*\pi$ . □

**Theorem 4.6** *Suppose  $Q\pi b \leq Qa\pi$ . The following rule is sound:*

$$\frac{Q \rightarrow [\pi]G \quad Q \rightarrow [a]Q}{Q \rightarrow [\pi]R}. \quad (4.12)$$

*Proof.* From the two premises, we have  $Q \rightarrow [\pi]G \wedge [a]Q$ , therefore

$$Q \rightarrow \nu X.[\pi]G \wedge [a]X = [a^*\pi]G.$$

By Lemmas 4.3 and 4.5,  $Q \rightarrow [\pi b^*]G$ , that is,  $Q \rightarrow [\pi]R$ . □

In most of our examples, the condition  $Q\pi b \leq Qa\pi$  and the premises of the rule (4.12) are satisfied. For example, for recursive types, the first premise says that if  $\sigma \leq \rho \leq \tau$ , and if  $(\sigma, \tau)$  has no  $b$ -successors, then either  $\sigma = \perp \vee \tau = \top$ . The second premise says that if

$$\sigma_1 \rightarrow \sigma_2 \leq \rho_1 \rightarrow \rho_2 \leq \tau_1 \rightarrow \tau_2,$$

then  $\tau_1 \leq \rho_1 \leq \sigma_1$  and  $\sigma_2 \leq \rho_2 \leq \tau_2$ . The condition  $Q\pi b \leq Qa\pi$  says that if

$$\sigma_1 \rightarrow \sigma_2 \leq \rho \leq \tau_1 \rightarrow \tau_2,$$

then  $\rho$  is of the form  $\rho_1 \rightarrow \rho_2$ .

## 5 Conclusion

We have introduced a new coinduction rule and illustrated its use in informal mathematical arguments with several examples involving more general statements than the usual equality proofs. The style of reasoning is similar to the use of induction.

An interesting research direction is to investigate whether a similar proof principle holds for properties and relations defined as least fixpoints. If this is indeed the case, can we also devise a mixed principle for induction and coinduction?

In the realm of metric coinduction, a similar proof principle has been proposed in [11]. Studying connections and possible generalizations of both proof principles will possibly encompass a change in category or a more categorical formulation of the principle. We would also like to explore whether we can incorporate other known proof techniques such as bisimulation up-to [16]. We leave these investigations for future work.

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