Left-Handed Completeness

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Abstract. We give a new, significantly shorter proof of the completeness of the left-handed star rule of Kleene algebra. The proof exposes the rich interaction of algebra and coalgebra in the theory of Kleene algebra.

1 Introduction

Axiomatizations of the equational theory of the regular sets over a finite alphabet have received much attention over the years. The topic was introduced in the seminal 1956 paper of Kleene [6], who left axiomatization as an open problem. Salomaa [15] gave two complete axiomatizations, but these depended on rules of inference that were sound under the standard interpretation but not under other natural interpretations. Conway, in his monograph [3], coined the term *Kleene algebra* (KA) and contributed substantially to the understanding of the question of axiomatization. An algebraic solution was presented by Kozen [9], who postulated two equational implications, similar to the inference rules of Salomaa; but unlike Salomaa's rules, they are universal Horn formulas, therefore sound over a variety of nonstandard interpretations. The main goal of this paper is to show that only one of the two implications is enough to guarantee completeness.

This result, which we shall call *left-handed completeness*, is a known result. It was claimed without proof by Conway [3, Theorem 12.5]. The only extant proof, by Boffa [1], relies on a lengthy (137 journal pages!) result of Krob [12], who presented a schematic equational axiomatization representing infinitely many equations. Krob's result was also later reworked and generalized in the framework of iteration theories [4].

Purely equational axiomatizations are undesirable for several reasons. From a practical point of view, they are inadequate for reasoning in the presence of other equational assumptions, which is almost always the case in real-life applications. For example, consider the redundant assignment $x := 1\,;\, x := 1$ and let a stand for $x := 1$. We have $aa = a$, since the assignment is redundant. We would expect this equation to imply $a^* = 1 + a$ (intuitively, performing the assignment $x := 1$ any number of times is equivalent to performing it zero or one times), but this is not entailed by the equational theory plus the extra equation $aa = a$. To see this, consider the free R-algebra (Conway's terminology for an algebra satisfying all

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the equations of the regular sets) on the finite monoid $\{1, a\}$, where $aa = a$. This algebra contains six elements: $0, 1, a, 1 + a, a^*, aa^*$. The elements a^* and $1 + a$ are distinct, even under the assumption $aa = a$, which is not at all desirable. This is an example of a finite algebra that satisfies all the equations of KA but is not a KA itself, because in a finite KA a star is always equal to a finite sum of powers. This example shows that purely equational axiomatizations would be inadequate for even the simplest verification tasks involving iteration in the presence of other equations.

On the other hand, characterizing a^* as a least fixpoint is a natural and powerful device, and is satisfied in virtually all models that arise in real life. However, there are interesting and useful models that satisfy only one of the two star rules [7,8,11], so it is useful to know that only one of the rules is needed for equational completeness.

Even though we present a new proof of a known result, there is added value in the exploration of the exquisite interplay between algebra and coalgebra in the theory of regular sets, which is visible throughout the technical development of the paper and notably in the novel definition of a *differential Kleene algebra*, which captures abstractly the relationship between the algebraic and coalgebraic structure of KA. The (syntactic) Brzozowski derivative provides the link from the algebraic to the coalgebraic view of regular expressions, whereas the canonical embedding of a given coalgebra into a matrix algebra plays the converse role. This interplay between algebra and coalgebra, first explored in [5,13], has opened the door to far-reaching extensions of Kleene's theorem and Kleene algebras [16].

Another contribution is a clear characterization of how far one can go in the proof of completeness with just equations. We show that the equational implication is needed only at two places (Lemmas 6 and 16). Furthermore, we show that the existence of least solutions implies uniqueness of solutions in the free algebra, which neatly ties our axiomatization with the original axiomatization of Salomaa.

Proofs omitted from the main text can be found in the extended version of this paper [10].

2 Axiomatization

2.1 Left-Handed Kleene Algebra

A *weak Kleene algebra* (*weak* KA) is an idempotent semiring with star satisfying (1)–(4):

$$a^* = 1 + aa^* \tag{1}$$

$$(ab)^*a = a(ba)^* \tag{2}$$

$$(a + b)^* = a^*(ba^*)^* \tag{3}$$

$$a^{**} = a^* \tag{4}$$

Axioms (2) and (3) are called *sliding* and *denesting*, respectively. These axioms were studied in depth by Conway [3] under the names *productstar* (for the combination of (1) and (2) in the single equation $(ab)^* = 1 + a(ba)^*b$), *sumstar*, and *starstar*, respectively. Although incomplete, these equations are sufficient for many arguments involving the star operator.

Conway studied many other useful families of axioms, including the *powerstar rules*

$$a^* = (a^n)^* \sum_{i=0}^{n-1} a^i, \tag{5}$$

although we will not need them here.

A *left-handed Kleene algebra* (LKA) is a weak KA satisfying a certain universal Horn formula, called the *left-handed star rule*, which may appear in either of the two equivalent forms

$$b + ax \leq x \Rightarrow a^*b \leq x \qquad\qquad ax \leq x \Rightarrow a^*x \leq x, \tag{6}$$

where $\leq$ is the natural partial order given by $a \leq b \Leftrightarrow a+b = b$. One consequence is the *left-handed bisimulation rule*

$$ax \leq xb \Rightarrow a^*x \leq xb^*. \tag{7}$$

2.2 Matrices

Let $\mathrm{Mat}(S, K)$ be the family of square matrices with rows and columns indexed by a finite set S with entries in a semiring K. Conway [3] shows that under the appropriately defined matrix operations, axioms (1)–(4) imply themselves for matrices. This is also true for (6) [9]. It is known for the powerstar rules (5) too, but only in a weaker form [3].

The *characteristic matrix* P_f of a function $f : S \to S$ has $(P_f)_{st} = 1$ if $f(s) = t$, 0 otherwise. A matrix is a *function matrix* if it is P_f for some f; that is, each row contains exactly one 1 and all other entries are 0.

Let $S_1, \ldots, S_n \subseteq S$ be a partition of S. A matrix $A \in \mathrm{Mat}(S, K)$ is said to be *block diagonal with blocks* $S_1, \ldots, S_n$ if $A_{st} = 0$ whenever s and t are in different blocks.

Lemma 1. *Let $A, P_f \in \mathrm{Mat}(S, K)$ with P_f the characteristic matrix of a function $f : S \to S$. The following are equivalent:*

(i) *A is block diagonal with blocks refining the kernel of f; that is, if $A_{st} \neq 0$, then $f(s) = f(t)$;*
(ii) *$AP_f = DP_f$ for some diagonal matrix D;*
(iii) *$AP_f = DP_f$, where D is the diagonal matrix $D_{ss} = \sum_{f(s)=f(t)} A_{st}$.*

2.3 Differential Kleene Algebra

A *differential Kleene algebra* (DKA) K is a weak KA containing a (finite) set $\Sigma \subseteq K$, called the *actions*, and a subalgebra C, called the *observations*, such that

(i) $ac = ca$ for all $a \in \Sigma$ and $c \in C$, and
(ii) C and Σ generate K,

and supporting a *Brzozowski derivative* consisting of a pair of functions $\varepsilon\colon K \to C$ and $\delta_a\colon K \to K$ for $a \in \Sigma$ satisfying the equations in Fig. 1. Thus $\varepsilon\colon K \to C$ is

$$
\begin{aligned}
\delta_a(e_1+e_2) &= \delta_a(e_1)+\delta_a(e_2) & \varepsilon(e_1+e_2) &= \varepsilon(e_1)+\varepsilon(e_2)\\
\delta_a(e_1e_2) &= \delta_a(e_1)e_2+\varepsilon(e_1)\delta_a(e_2) & \varepsilon(e_1e_2) &= \varepsilon(e_1)\varepsilon(e_2)\\
\delta_a(e^*) &= \varepsilon(e^*)\delta_a(e)\,e^* & \varepsilon(e^*) &= \varepsilon(e)^*\\
\delta_a(b) &= \begin{cases} 1 & \text{if } a=b,\\ 0 & \text{if } a\neq b,\end{cases} \quad b\in\Sigma & \varepsilon(b) &= 0,\ b\in\Sigma\\
\delta_a(c) &= 0,\ c\in C & \varepsilon(c) &= c,\ c\in C
\end{aligned}
$$

Fig. 1: Brzozowski derivatives

a retract (a KA homomorphism that is the identity on C, which immediately implies $0, 1 \in C$). The functions δ_a and ε impart a coalgebra structure of signature $-^{\Sigma} \times C$ in addition to the Kleene algebra structure.

This definition is a slight generalization of the usual situation in which $C = \mathbb{2} = \{0, 1\}$ and the function ε and δ_a are the (syntactic) Brzozowski derivatives. We will be primarily interested in matrix KAs in which C is the set of square matrices over $\mathbb{2}$.

2.4 Examples

One example of a DKA with observations $\mathbb{2}$ is $\mathsf{Brz} = (2^{\Sigma^*}, \delta, \varepsilon)$, where $\varepsilon(A) = 1$ iff A contains the null string and 0 otherwise, and $\delta_a : 2^{\Sigma^*} \to 2^{\Sigma^*}$ is the classical *Brzozowski derivative*

$$\delta_a(A) = \{x \in \Sigma^* \mid ax \in A\}.$$

This is the final coalgebra of the functor $-^{\Sigma} \times \mathbb{2}$ [13]. It is also an LKA under the usual set-theoretic operations.

Another example is the free LKA K_Σ on generators Σ. It is also a DKA, where δ_a and ε are defined inductively on the syntax of regular expressions according to Fig. 1. The maps δ_a and ε are easily shown to be well defined modulo the axioms of LKA.

These structures possess both an algebra and a coalgebra structure, and in fact are bialgebras [5]. Our main result essentially shows that the latter is isomorphically embedded in the former.

2.5 Properties of DKAs

Silva [16] calls the following result the *fundamental theorem* in analogy to a similar result proved for infinite streams by Rutten [14], closely related to the fundamental theorem of calculus. It is fundamental in the sense that it connects the differential structure, given by δ_a and ε, with the axioms of LKA. We show here that the result holds under weaker assumptions than those assumed in [16]: in fact, we prove this theorem only using equations.

Theorem 1. *Let K be a DKA. For all elements $e \in K$,*

$$e = \sum_{a \in \Sigma} a\delta_a(e) + \varepsilon(e). \tag{8}$$

Proof. We proceed by induction on the generation of e from Σ and C using only equations of weak KA and properties of derivatives. For $e \in C$, $\varepsilon(e) = e$ and $\delta_a(e) = 0$, thus (8) holds. For $e = a \in \Sigma$, the right-hand side of (8) reduces to a, thus (8) holds in this case as well.

For the induction step, the case of $+$ is straightforward. For multiplication,

$$\begin{aligned} e_1e_2 &= (\sum_{a \in \Sigma} a\delta_a(e_1) + \varepsilon(e_1))e_2 = \sum_{a \in \Sigma} a\delta_a(e_1)e_2 + \varepsilon(e_1)(\sum_{a \in \Sigma} a\delta_a(e_2) + \varepsilon(e_2)) \\ &= \sum_{a \in \Sigma} a\delta_a(e_1)e_2 + \sum_{a \in \Sigma} a\varepsilon(e_1)\delta_a(e_2) + \varepsilon(e_1)\varepsilon(e_2) \\ &= \sum_{a \in \Sigma} a(\delta_a(e_1)e_2 + \varepsilon(e_1)\delta_a(e_2)) + \varepsilon(e_1e_2) = \sum_{a \in \Sigma} a\delta_a(e_1e_2) + \varepsilon(e_1e_2). \end{aligned}$$

For e^*, we use the KA identity

$$(x + y)^* = y^*x(x + y)^* + y^*, \tag{9}$$

which follows equationally from (1), (3), and distributivity. Using this identity with $x = \sum_{a \in \Sigma} a\delta_a(e)$ and $y = \varepsilon(e)$,

$$\begin{aligned} e^* &= (\sum_{a \in \Sigma} a\delta_a(e) + \varepsilon(e))^* = \varepsilon(e)^* \sum_{a \in \Sigma} a\delta_a(e)e^* + \varepsilon(e)^* \qquad \text{by (9)} \\ &= \sum_{a \in \Sigma} a\varepsilon(e)^*\delta_a(e)e^* + \varepsilon(e)^* = \sum_{a \in \Sigma} a\delta_a(e^*) + \varepsilon(e^*). \end{aligned}$$

□

Let K be a DKA with actions Σ and observations C. We define the *C-free part* of $e \in K$ to be

$$e' = \sum_{a \in \Sigma} a\delta_a(e). \tag{10}$$

By the fundamental theorem, every element of K can be decomposed into its C-free part e' and $\varepsilon(e) \in C$.

$$e = e' + \varepsilon(e) \qquad\qquad \varepsilon(e') = 0. \tag{11}$$

The map $e \mapsto e'$ is linear and satisfies properties akin to *derivations* in calculus:

$$1' = 0 \qquad (de)' = d'e + de' \qquad {e^*}' = \varepsilon(e^*)(e' \cdot \varepsilon(e^*))^+. \tag{12}$$

For the last two,

$$\begin{aligned}
\sum_{a\in\Sigma} a\delta_a(de) &= \sum_{a\in\Sigma} a\delta_a(d)e + \sum_{a\in\Sigma} a\varepsilon(d)\delta_a(e)\\
&= d'e + \varepsilon(d)e' = d'e' + d'\varepsilon(e) + \varepsilon(d)e' = d'e + de',
\end{aligned}$$

$$\begin{aligned}
\sum_{a\in\Sigma} a\delta_a(e^*) &= \sum_{a\in\Sigma} a\varepsilon(e^*)\delta_a(e)e^* = \varepsilon(e^*)\left(\sum_{a\in\Sigma} a\delta_a(e)\right)e^*\\
&= \varepsilon(e^*)e'e^* = \varepsilon(e^*)e'(e' + \varepsilon(e))^*\\
&= \varepsilon(e^*)e'\varepsilon(e)^*(e'\varepsilon(e)^*)^* = \varepsilon(e^*)(e' \cdot \varepsilon(e^*))^+.
\end{aligned}$$

Note that for $C = \mathbb{2}$, the rightmost equation in (12) simplifies to ${e^*}' = {e'}^+$, using the fact that $\varepsilon(e^*) = 1$.

Moreover, the decomposition is unique: if $e = b + c$ with $\varepsilon(b) = 0$ and $c' = 0$, then

$$b = b' + \varepsilon(b) = b' + c' = e' \qquad c = c' + \varepsilon(c) = \varepsilon(b) + \varepsilon(c) = \varepsilon(e).$$

The following consequences of the above observations will be useful in our application (more precisely in Lemma 13). If $G \subseteq K$, let $\langle G\rangle$ denote the subalgebra of K generated by G.

Lemma 2. *Let K be a* DKA *with derivation $'$. Let $G \subseteq K$ and $x \in K$. If $e' = e'x$ and $\varepsilon(e) \in \mathbb{2}$ for all $e \in G$, then $e' = e'x$ and $\varepsilon(e) \in \mathbb{2}$ for all $e \in \langle G\rangle$.*

Proof. We have $1'x = 0'x = 0$ and $e'x = e'$ for $e \in G$, and by induction,

$$\begin{aligned}
(d+e)'x &= d'x + e'x = d' + e' = (d+e)',\\
(de)'x &= d'ex + de'x = d'e'x + d'\varepsilon(e)x + de'x = d'e + de' = (de)',\\
(e^*)'x &= {e'}^+x = {e'}^*e'x = {e'}^*e' = {e'}^+ = {e^*}'.
\end{aligned}$$

Also, $\varepsilon(e) \in \mathbb{2}$ for all $e \in \langle G\rangle$ because ε is a homomorphism. □

Lemma 3. *Let K be a* DKA *with derivation $'$. Suppose $G \subseteq K$ and $x, x^- \in C$ such that $x^-x = 1$ and $e'xx^- = e'$ and $\varepsilon(e) \in \mathbb{2}$ for all $e \in G$. Then the map $e \mapsto x^-ex$ is a* KA *homomorphism on $\langle G\rangle$.*

Proof. It is clearly a homomorphism with respect to 0, 1, and +. By Lemma 2, we can assume that $e'xx^- = e'$ and $\varepsilon(e) \in \mathbb{2}$ for all $e \in \langle G \rangle$. Now to show that the map preserves multiplication and star,

$$\begin{aligned}
x^- dex &= x^-(d' + \varepsilon(d))ex = x^- d'ex + x^- \varepsilon(d)ex = x^- d'xx^- ex + x^- xx^- \varepsilon(d)ex \\
&= x^- d'xx^- ex + x^- \varepsilon(d)xx^- ex = x^- dxx^- ex, \\
x^- e^* x &= x^-(e' + \varepsilon(e))^* x = x^- e'^* x = x^-(e'xx^-)^* x \\
&= (x^- e'x)^* x^- x = (x^- e'x + x^- x\varepsilon(e))^* = (x^-(e' + \varepsilon(e))x)^* = (x^- ex)^*.
\end{aligned}$$

Above, we use the fact that $\varepsilon(d)f = f\varepsilon(d)$, for any expression f, since $\varepsilon(d) \in \mathbb{2}$. ☐

2.6 Systems of Linear Equations

A *system of (left-)linear equations* over a weak KA K is a coalgebra (S, D, E) of signature $-^{\Sigma} \times K$, where $\Sigma \subseteq K$, $D\colon S \to S^{\Sigma}$, and $E\colon S \to K$. A finite system corresponds to a finite coalgebra, that is the set of states S is finite. We curry D so as to write $D_a\colon S \to S$ for $a \in \Sigma$. The map $D\colon \Sigma \to S \to S$ extends uniquely to a monoid homomorphism $D\colon \Sigma^* \to S \to S$, thus we have $D_x : S \to S$ for $x \in \Sigma^*$. A *solution* in K is a map $\phi : S \to K$ such that

$$\phi(s) = \sum_{a \in \Sigma} a\phi(D_a(s)) + E(s). \tag{13}$$

Every finite system of linear equations has a solution. To see this, form an associated matrix $A \in \mathrm{Mat}(S, K)$, where

$$A = \sum_{a \in \Sigma} \Delta(a)P(a) \in \mathrm{Mat}(S, K),$$

where $\Delta(a)$ is the diagonal matrix with diagonal entries a and $P(a)$ is the characteristic matrix of the function D_a. Regarding ϕ and E as column vectors indexed by S, the solution condition (13) takes the form $\phi = A\phi + E$. Since $\mathrm{Mat}(S, K)$ is a weak KA, the vector A^*E is a solution by (1). We call this solution the *canonical solution*. If in addition K is an LKA, then the canonical solution is also the least solution.

If K is freely generated by Σ, then the map $a \mapsto \Delta(a)P(a)$ extends uniquely to a KA homomorphism $\chi : K \to \mathrm{Mat}(S, K)$, called the *standard embedding*. It will follow from our results that χ is injective.

3 Decompositions

3.1 Simple Strings

Let (S, D, E) be a finite coalgebra of type $-^{\Sigma} \times \mathbb{2}$. Let K_{Σ} be the free LKA on generators Σ. Extend D to a monoid homomorphism $D : \Sigma^* \to S \to S$.

The corresponding characteristic matrices P also extend homomorphically by matrix multiplication. Let $\chi\colon K_\Sigma \to \mathrm{Mat}(S, K_\Sigma)$ with $\chi(a) = \Delta(a)P(a)$ be the standard embedding as defined in Section 2.6.

Call $x \in \Sigma^*$ *simple* if $P(y) \neq P(z)$ for all distinct suffixes y, z of x. If x is simple, then so are all its suffixes. Define

$$M = \{x \mid x \text{ is simple}\}$$
$$M_x = \{y \mid |y| > 0 \text{ and } P(yx) = P(x), \text{ but all proper suffixes of } yx \text{ are simple}\}.$$

Every string can be reduced to a simple string by repeatedly removing certain substrings while preserving P(-). This is the well-known *pumping lemma* from automata theory. If y is not simple, find a suffix vw such that $P(vw) = P(w)$ and $v \neq \varepsilon$, and remove v. The resulting string is shorter and P(-) is preserved. Repeating this step eventually produces a string $x \in M$ such that $P(y) = P(x)$. If we always choose the shortest eligible suffix vw, so that $v \in M_w$—this strategy is called *right-to-left greedy*—we obtain a particular element $\gamma(y) \in M$ related to the construction of V_y (see Lemma 7).

Let $n = |S|$. If $y \in M_x$, then $1 + |x| \leq |yx| \leq n^n$, as each function $S \to S$ is represented at most once as $P(z)$ for a proper suffix z of yx.

We now define a family of elements R_x, $T_{y,x}$, and V_x of K_Σ for $x, y \in \Sigma^*$.

$$R_x = \left(\sum_{y\in M_x} T_{y,x}\right)^* \qquad T_{1,x} = 1 \qquad T_{ay,x} = R_{ayx}aT_{y,x},\ a \in \Sigma \tag{14}$$

$$V_x = T_{x,1}R_1 \qquad\qquad V = \sum_{x\in M} V_x. \tag{15}$$

Intuitively, if x is a simple word labeling a path from s to t, then all words represented by V_x lead from s to t, and V represents all words in Σ^*. The expressions R_x and $T_{y,x}$ allow the encoding of loops.

The definitions of R_x and $T_{y,x}$ in (14) are by mutual induction, but it is not immediately clear that the definition is well-founded: note that R_x depends on $T_{y,x}$ for $y \in M_x$, which depends on R_{yx}. To prove well-foundedness, we define a binary relation $\succ$ on tuples (R, x) and (T, y, x) defined as follows. For $x, y \in \Sigma^*$ and $a \in \Sigma$, let

$$(R, x) \succ (T, y, x),\ y \in M_x \qquad (T, ay, x) \succ (R, ayx) \qquad (T, ay, x) \succ (T, y, x).$$

The relation $\succ$ describes the dependencies in the definition (14).

Lemma 4. *The relation $\succ$ is well-founded; that is, there are no infinite $\succ$-paths.*

Note that $R_x = 1$ for $|x| \geq n^n$, since the sum in the definition of R_x in (14) is vacuous in that case. It follows inductively that $T_{y,x} = y$ for $|x| \geq n^n$.

Lemma 5. *For all $x, y \in \Sigma^*$ and $a \in \Sigma$,*

(i) $V_1 = R_1$ *and* $V_{ax} = R_{ax}aV_x$.

(ii) $V_{yx} = T_{y,x} V_x$.

Proof. For (i), we have $V_1 = T_{1,1} R_1 = R_1$ and $V_{ax} = T_{ax,1} R_1 = R_{ax} a T_{x,1} R_1 = R_{ax} a V_x$. For (ii), we proceed by induction on $|y|$. The basis $V_x = T_{1,x} V_x$ is immediate. For the induction step, using (i), $V_{ayx} = R_{ayx} a V_{yx} = R_{ayx} a T_{y,x} V_x = T_{ay,x} V_x$. □

In the following two lemmas, we will exploit the fact that $R_z V_z = V_z$, which can be proven by case analysis on z using the fact that $R_z R_z = R_z$.

Lemma 6. $\left(\sum_{a \in \Sigma} a\right)^* = V$.

Proof. For the forward inequality, we use the left-handed star rule (6). Let $x \in M$ and $a \in \Sigma$. By Lemma 5(i), $aV_x \le R_{ax} a V_x = V_{ax}$. If $ax \in M$, then $V_{ax} \le V$. If $ax \notin M$, say $x = yz$ with $P(ax) = P(ayz) = P(z)$, then $ay \in M_z$ and $z \in M$. By Lemma 5, $V_{ax} = V_{ayz} = T_{ay,z} V_z \le R_z V_z = V_z \le V$. In either case, $aV_x \le V$. Since $a \in \Sigma$ and $x \in M$ were arbitrary, $(\sum_{a \in \Sigma} a) V \le V$. Also $1 \le V$, since $1 \le R_1 = V_1$. By (6), $(\sum_{a \in \Sigma} a)^* \le V$. The reverse inequality follows from monotonicity. □

Lemma 7. *For all $y \in \Sigma^*$, $V_y \le V_{\gamma(y)}$.*

Proof. If $v \in M_w$, then, by Lemma 5(ii), we have that $V_{vw} = T_{v,w} V_w \le V_w$, since $T_{v,w} \le R_w$ and $R_w V_w \le V_w$. The result follows inductively from the right-to-left construction of $\gamma(y)$. □

3.2 Decompositions

Let (S, D, E) be a finite coalgebra of type $-^{\Sigma} \times 2$ with standard embedding

$$\chi : K_\Sigma \to \mathrm{Mat}(S, K_\Sigma) \qquad\qquad \chi(a) = \Delta(a) P(a).$$

Let $e \in K_\Sigma$. A *decomposition* of e (with respect to χ) is a family of expressions $e_x \in K_\Sigma$ indexed by $x \in M$ (recall that M is the set of simple strings) such that

(a) $e = \sum_x e_x$, and
(b) $\chi(e_x) = \Delta(e_x) P(x)$ for all $x \in M$.

It follows that

$$\chi(e) = \sum_x \Delta(e_x) P(x). \tag{16}$$

If P, Q are matrices, we say that the decomposition *respects* P, Q if in addition

(c) $P(x) Q = P$ for all x such that $e_x \neq 0$.

We say that e is *decomposable* if it has a decomposition. We will eventually show that all expressions are decomposable.

Lemma 8. *Let $x \mapsto e_x$ be a decomposition of e. The decomposition respects P, Q iff $\chi(e)Q = \Delta(e)P$.*

Proof. If the decomposition respects P, Q, then

$$\chi(e)Q = \sum_x \Delta(e_x)P(x)Q = \sum_x \Delta(e_x)P = \Delta(\sum_x e_x)P = \Delta(e)P.$$

Conversely, if $e_x \neq 0$ and $P(x)Q \neq P$, then $\Delta(e_x)P(x)Q \neq \Delta(e)P$, therefore

$$\chi(e)Q = \sum_x \Delta(e_x)P(x)Q \neq \Delta(e)P.$$

□

We have specified the index set M in the definition of decomposition to emphasize that the $P(x)$ must be generated by the $P(a)$, but in fact any finite index set will do, provided the function matrices are so generated.

Lemma 9. *Let e_α and P_α be finite indexed collections of elements of K_Σ and function matrices, respectively, such that*

$$e = \sum_\alpha e_\alpha \qquad\qquad \chi(e_\alpha) = \Delta(e_\alpha)P_\alpha$$

and such that each P_α is $P(y_\alpha)$ for some $y_\alpha \in \Sigma^*$. Then $e_x = \sum_{x=\gamma(y_\alpha)} e_\alpha$ is a decomposition of e.

Proof. By Lemma 7, if $x = \gamma(y_\alpha)$, then $P(x) = P(y_\alpha)$. Easy calculations then show $e = \sum_x e_x$ and $\chi(e_x) = \Delta(e_x)P(x)$. □

Decompositions can be combined additively or multiplicatively. The *sum* and *product* of two decompositions $F : M \to K_\Sigma$ and $G : M \to K_\Sigma$ are, respectively, the decompositions

$$(F + G)(x) = F(x) + G(x) \qquad (F \times G)(x) = \sum_{x=\gamma(yz)} F(y)G(z).$$

Lemma 10.

(i) *If F is a decomposition of e and G is a decomposition of d, then $F + G$ is a decomposition of $e + d$. If F and G both respect P, Q, then so does $F + G$.*
(ii) *If F is a decomposition of e and G is a decomposition of d, then $F \times G$ is a decomposition of ed. If F respects P, Q and G respects Q, R, then $F \times G$ respects P, R.*

To handle star, we describe a monad structure on systems built on top of the string monad. The motivation is that we wish to consider the elements of M as single letters of an alphabet. To avoid confusion, we use $\alpha, \beta, \ldots$ to denote words in M^*. In §2.6, we constructed the standard embedding χ with respect to a coalgebra (S, D, E) of type $-^\Sigma \times C$. Now we wish to do the same for the alphabet M. We thus have a coalgebra $(S, \hat{D})$ with $\hat{D}_x : S \to S$ of type $-^M$ with $\hat{D}_x = D_x$. The only difference is that on the left-hand side, x is considered as a single letter, whereas on the right-hand side, D_x is defined inductively from D_a for $a \in \Sigma$. The standard embedding is η, defined in the same way for (S, M) as χ was defined for (S, D):

$$\eta : K_M \to \mathrm{Mat}(S, K_M) \qquad\qquad \eta(x) = \Delta(x)P(x),\ x \in M.$$

Now let $\hat{M}$ be constructed as in §3.1 for the alphabet M as M was constructed for Σ.

Lemma 11. *Suppose that $(\sum_{x\in M} x)^* \in K_M$ has a decomposition d_α, $\alpha \in \hat{M}$ with respect to η and that $e \in K_\Sigma$ has a decomposition $\sigma : x \mapsto e_x$ with respect to χ. Let $\mu(x) = \sum_{x=\gamma(\alpha)} d_\alpha$. Then $\sigma\mu : x \mapsto \sigma(\sum_{x=\gamma(\alpha)} d_\alpha)$ is a decomposition of e^* with respect to χ. Moreover, if the decomposition of e respects Q, Q, then so does the decomposition e^*.*

3.3 Existence of Decompositions

Let (S, D, E) be a finite coalgebra of type $-^\Sigma \times C$ with standard embedding $\chi\colon K_\Sigma \to \mathrm{Mat}(S, K_\Sigma)$. Let $M \subseteq \Sigma^*$ and $M_x \subseteq \Sigma^*$ for $x \in M$ be defined as in §3.1. Let R_x, $T_{y,x}$, and $V_x \in K_\Sigma$ be as defined in §3.1 with respect to M and M_x.

In the following, the term *decomposition* refers to decompositions with respect to χ. A *universal decomposition* is a decomposition for the universal expression $(\sum_{a\in\Sigma} a)^*$.

We remark that Lemmas 12 and 13 are actually co-dependent and require proof by mutual induction on the well-founded relation $\succ$ and on dimension of the associated matrices. Lemma 12 can be proved for permutations without reference to Lemma 13 (this is the basis of the induction), but in the general case requires Lemma 13 for lower dimension; and the proof of Lemma 13 depends on Lemma 12 for permutations.

Lemma 12. *For $x, y \in \Sigma^*$,*

(i) *$T_{y,x}$ has a decomposition respecting $P(yx), P(x)$.*
(ii) *R_x has a decomposition respecting $P(x), P(x)$.*
(iii) *$x \mapsto V_x$ is a universal decomposition.*

Proof. The proof is by induction on the well-founded relation $\succ$, using the fact that χ and Δ are homomorphisms, and on dimension. Let us assume that the lemma is true for all matrices of smaller dimension.

For (i), $T_{1,x} = 1$ has the trivial decomposition $1 \mapsto 1$ and $x \mapsto 0$ for all $x \in M - \{1\}$, and this clearly respects $P(x), P(x)$.

For ay, we have $T_{ay,x} = R_{ayx} a T_{y,x}$. By the induction hypothesis, we have a decomposition for R_{ayx} respecting $P(ayx), P(ayx)$ and a decomposition for $T_{y,x}$ respecting $P(yx), P(x)$. We also have the trivial decomposition $a \mapsto a$ and $x \mapsto 0$

for all $x \in M - \{a\}$, which respects $P(ayx), P(yx)$. By Lemma 10(ii), the product of these three decompositions in the appropriate order is a decomposition for $T_{ay,x}$ respecting $P(ayx), P(x)$.

For (ii), we have $R_x = e^*$, where $e = \sum_{y \in M_x} T_{y,x}$. By the induction hypothesis, we can assume decompositions of $T_{y,x}$ for each $y \in M_x$ respecting $P(yx), P(x)$. Since $P(yx) = P(x)$ for $y \in M_x$, these decompositions also respect $P(x), P(x)$. By Lemma 10(i), the sum of these decompositions gives a decomposition of e respecting $P(x), P(x)$. By Lemma 8, $\chi(e)P(x) = \Delta(e)P(x)$.

If $P(x)$ is invertible, then $\chi(e) = \Delta(e)$, therefore

$$\chi(R_x) = \chi(e)^* = \Delta(e)^* = \Delta(R_x).$$

In this case, we can decompose R_x trivially as $1 \mapsto R_x$ and $y \mapsto 0$ for $y \in M - \{1\}$, which respects $P(x), P(x)$, and we are done.

If $P(x)$ is not invertible, we can use Lemma 13 to reduce the problem to a lower dimension. By that lemma, we have a universal decomposition that we can use with Lemma 11 to obtain a decomposition of e^* respecting $P(x), P(x)$.

For (iii),

$$\begin{aligned}\chi(V_x) &= \chi(T_{x,1})\chi(R_1) = \chi(T_{x,1})P(1)\chi(R_1) = \Delta(T_{x,1})P(x)\Delta(R_1) \\ &= \Delta(T_{x,1}R_1)P(x) = \Delta(V_x)P(x).\end{aligned}$$

Combined with Lemma 6, this makes $x \mapsto V_x$ a universal decomposition. $\square$

Lemma 13. *There exists a universal decomposition.*

Proof. The proof is by induction on dimension and on the number of letters of Σ. We can assume by Lemma 12 that we already have a universal decomposition for the subalphabet of Σ consisting of all a such that $P(a)$ is invertible. Now we show how to add in the rest of the elements of Σ one by one.

Suppose we have constructed a universal decomposition $x \mapsto e_x$ for a subalphabet $\Gamma \subseteq \Sigma$ including all a such that $P(a)$ is invertible. Let $e = \sum_{a \in \Gamma} a$ and $a \in \Sigma - \Gamma$. We have

$$e^* = \sum_x e_x \qquad\qquad \chi(e^*) = \Delta(e_x)P(x),$$

and we wish now to construct a decomposition for $(a+e)^*$.

Since $P(a)$ is not a permutation, the range of the corresponding function is a proper subset $C \subset S$. Equivalently stated, the $S \times (S - C)$ submatrix of $P(a)$ is the zero matrix. Let X be the $S \times C$ matrix whose $C \times C$ submatrix is the identity matrix and whose other entries are 0, and let X^T be its transpose. The following facts are easy to verify:

$$P(a) = P(a)XX^T \qquad\qquad X^T X = I. \tag{17}$$

These are square matrices of dimension $S \times S$ and $C \times C$, respectively. Now

$$(a+e)^* = (e^*a)^*e^* = (1 + e^*a(e^*a)^*)e^*.$$

By Lemma 10, we know how to combine decompositions additively and multiplicatively, and we have decompositions of a, e^*, and 1. It thus suffices to construct a decomposition of $a(e^*a)^*$.

We can reduce to a lower dimensional $C \times C$ problem. Let

$$R(x) = XX^T P(xa) \qquad\qquad Q(x) = X^T P(xa)X.$$

The matrix $R(x)$ is the $S \times S$ matrix whose $C \times C$ submatrix is $Q(x)$ and whose other entries are 0. It follows from (17) that

$$R(x) = XQ(x)X^T \qquad\qquad R(\alpha) = XQ(\alpha)X^T \tag{18}$$

for any $\alpha \in M^*$. Now consider the system

$$\eta : K_M \to \mathrm{Mat}(C, K_M) \qquad\qquad \eta(x) = \Delta(x)Q(x)$$

of dimension $C \times C$. By the induction hypothesis on dimension, we have a universal decomposition with respect to η:

$$(\sum_x x)^* = \sum_\alpha d_\alpha \qquad\qquad \eta(d_\alpha) = \Delta(d_\alpha)Q(\alpha)$$

where α ranges over $\hat{M}$. Let

$$P_\alpha = P(a)R(\alpha),\ \alpha \in \hat{M} \qquad\qquad \sigma(x) = e_x a.$$

The map σ extends uniquely to a KA homomorphism $\sigma : K_M \to K_\Sigma$. We claim that $a\sigma(d_\alpha)$ and P_α form a decomposition of $a(e^*a)^*$ with respect to χ. We must show that

$$a(e^*a)^* = \sum_\alpha a\sigma(d_\alpha) \qquad\qquad \chi(a\sigma(d_\alpha)) = \Delta(a\sigma(d_\alpha))P_\alpha. \tag{19}$$

According to Lemma 9, we must also show that the P_α are generated by the $P(a)$, $a \in \Sigma$. The left-hand equation of (19) is a straightforward calculation:

$$a(e^*a)^* = a(\sum_x e_x a)^* = a\sigma((\sum_x x)^*) = a\sigma(\sum_\alpha d_\alpha) = \sum_\alpha a\sigma(d_\alpha).$$

That the P_α are generated by the $P(a)$ can be shown inductively using (17):

$$\begin{aligned} P_1 &= P(a)R(1) = P(a)XX^T P(a) = P(a^2) \\ P_{x\alpha} &= P(a)R(x)R(\alpha) = P(a)XX^T P(xa)R(\alpha) = P(ax)P(a)R(\alpha) = P(ax)P_\alpha. \end{aligned}$$

It remains to prove the right-hand equation of (19). Let G be the image of the map $\chi\sigma : K_M \to \mathrm{Mat}(S, K_\Sigma)$ defined by

$$\chi\sigma(x) = \chi(e_x a) = \Delta(e_x a)P(xa).$$

The generators satisfy $\chi\sigma(x)' = \chi\sigma(x)'XX^T$, so by Lemma 2, this also holds true for all elements of G, and $\varepsilon(A) \in \{0, I\}$ for all $A \in G$. Also, by Lemma 3, the map

$$A \mapsto X^T AX : G \to \mathrm{Mat}(C, K_\Sigma)$$

is a homomorphism on G, therefore so is its composition with $\chi\sigma$, the map $X^T(\chi\sigma)X : K_M \to \mathrm{Mat}(C, K_\Sigma)$.

Now $X^T(\chi\sigma)X = \hat{\sigma}\eta$, as they are both homomorphisms $K_M \to \mathrm{Mat}(C, K_\Sigma)$ and agree on the generators $x \in M$:

$$\begin{aligned}(X^T(\chi\sigma)X)(x) &= X^T(\chi\sigma(x))X = X^T(\chi(e_x a))X \\ &= X^T(\Delta(e_x a)P(xa))X = \Delta(e_x a)X^T P(xa)X = \Delta(e_x a)Q(x)\end{aligned}$$

$$\hat{\sigma}\eta(x) = \hat{\sigma}(\Delta(x)Q(x)) = \Delta(\sigma(x))Q(x) = \Delta(e_x a)Q(x).$$

Thus the value they take on $d_\alpha \in K_M$ is the same:

$$X^T\chi(\sigma(d_\alpha))X = \hat{\sigma}\eta(d_\alpha) = \hat{\sigma}(\Delta(d_\alpha)Q(\alpha)) = \Delta(\sigma(d_\alpha))Q(\alpha). \tag{20}$$

Calculating, we find

$$\begin{aligned}\chi(a\sigma(d_\alpha)) &= \chi(a\sigma(d_\alpha))' && \text{since } \varepsilon(\chi(a\sigma(d_\alpha))) = 0 \\ &= \Delta(a)P(a)XX^T\chi(\sigma(d_\alpha))XX^T && \text{by (17) and Lemma 2} \\ &= \Delta(a)P(a)X\Delta(\sigma(d_\alpha))Q(\alpha)X^T && \text{by (20)} \\ &= \Delta(a)\Delta(\sigma(d_\alpha))P(a)XQ(\alpha)X^T && \\ &= \Delta(a\sigma(d_\alpha))P(a)R(\alpha) && \text{by (18)} \\ &= \Delta(a\sigma(d_\alpha))P_\alpha && \text{by definition of } P_\alpha.\end{aligned}$$

□

Corollary 1. *All expressions are decomposable.*

Proof. We proceed by induction on structure of the expression. Every element $a \in \{0, 1\} \cup \Sigma$ has a trivial decomposition $1 \mapsto a$ and $x \mapsto 0$ for $x \in M - \{1\}$. Closure under sum and product follow from Lemma 10. For star, suppose we have a decomposition e_x, $x \in M$, of e. By Lemma 13, we have a decomposition for the universal expression $(\sum_{x \in M} x)^*$. Lemma 11 then provides a decomposition for e^* via the substitution $x \mapsto e_x$. □

4 Completeness

Let (S, D, E) be a coalgebra of signature $-^\Sigma \times 2$. We say that states $s, t \in S$ are *bisimilar*, and write $s \approx t$, if $E(D_x(s)) = E(D_x(t))$ for all $x \in \Sigma^*$. The relation $\approx$ is the maximal bisimulation on S and is the kernel of the unique coalgebra homomorphism $L_S : S \to \mathsf{Brz}$, where

$$L_S(s) = \{x \in \Sigma^* \mid E(D_x(s)) = 1\}.$$

Soundness and completeness can be expressed in the following terms. Let $\mathcal{E}$ be a set of equations or equational implications on regular expressions, and let $\mathsf{Con}\,(\mathcal{E})$ be the set of consequences of $\mathcal{E}$ in ordinary equational logic. The axioms $\mathcal{E}$ are *sound* if $\mathsf{Con}\,(\mathcal{E})$ refines bisimilarity; equivalently, if the Brzozowski derivative is well-defined on the free weak KA modulo $\mathcal{E}$. A sound set of axioms is *complete* if $\mathsf{Con}\,(\mathcal{E})$ and bisimilarity coincide; that is, if the unique coalgebra homomorphism to the final coalgebra Brz is injective. We have mentioned above that the LKA axioms are sound; indeed, soundness has been shown in [9] for a larger set of axioms, namely those of KA. To prove that they are complete, our task is to show that the unique coalgebra homomorphism $L_{K_\Sigma}\colon K_\Sigma \to \mathsf{Brz}$ is injective.

This characterization of soundness and completeness was first observed by Jacobs [5] for classical regular expressions and KA and largely explored in the thesis of Silva [16] for generalized regular expressions. See [16] for a comprehensive introduction to this characterization.

In what follows, recall that for a coalgebra (S, D, E) we form an associated matrix $A \in \mathrm{Mat}(S, K)$, where

$$A = \sum_{a\in\Sigma} \Delta(a)P(a) \in \mathrm{Mat}(S, K),$$

where $\Delta(a)$ is the diagonal matrix with diagonal entries a and $P(a)$ is the characteristic matrix of the function D_a.

Lemma 14. *If $s \approx t$ then $(A^*E)_s = (A^*E)_t$.*

Proof. We have

$$A = \sum_{a\in\Sigma} \Delta(a)P(a) = \sum_{a\in\Sigma} \chi(a) = \chi(\sum_{a\in\Sigma} a),$$

$$A^* = \chi(\sum_{a\in\Sigma} a)^* = \chi((\sum_{a\in\Sigma} a)^*) = \chi(\sum_{x\in M} V_x) = \sum_{x\in M} \chi(V_x) = \sum_{x\in M} \Delta(V_x)P(x).$$

Now for any $s \in S$,

$$\begin{aligned}(A^*E)_s &= (\sum_{x\in M} \Delta(V_x)P(x)E)_s = \sum_{x\in M} V_x(P(x)E)_s \\ &= \sum_{x\in M} V_x \sum_{u\in S} P(x)_{su}E_u = \sum_{x\in M} V_x E(D_x(s)).\end{aligned}$$

If $s \approx t$, then $E(D_x(s)) = E(D_x(t))$ for all $x \in \Sigma^*$, therefore

$$(A^*E)_s = \sum_{x\in M} V_x E(D_x(s)) = \sum_{x\in M} V_x E(D_x(t)) = (A^*E)_t.$$

□

Consider a finite subcoalgebra (S, δ, ε) of K_Σ, where δ and ε comprise the Brzozowski derivative as defined as in Fig. 1. Recall that every $e \in K_\Sigma$ generates a finite subcoalgebra, since it has finitely[3] many Brzozowski derivatives [13]. Let $\chi \colon K_\Sigma \to \mathrm{Mat}(S, K_\Sigma)$ be the standard embedding as defined in §2.6.

Lemma 15. $e = (\chi(e)E)_e$.

Proof. If $e_x \neq 0$, then there exists $y \in \Sigma^*$ such that $y \leq e_x$. Since χ is monotone,

$$\Delta(y)P(y) = \chi(y) \leq \chi(e_x) = \Delta(e_x)P(x),$$

therefore $P(y) = P(x)$. Moreover, $1 \leq \delta_y(e_x) \leq \delta_y(e)$, therefore $\varepsilon(\delta_y(e)) = 1$. Since $P(y) = P(x)$, $\varepsilon(\delta_x(e)) = 1$.

We have shown that if $e_x \neq 0$, then $\varepsilon(\delta_x(e)) = 1$; in other words, $e_x = e_x\varepsilon(\delta_x(e))$. It follows that

$$(\chi(e)E)_e = (\sum_x \Delta(e_x)P(x)E)_e = \sum_x e_x(P(x)E)_e = \sum_x e_x\varepsilon(\delta_x(e)) = \sum_x e_x = e.$$

□

Lemma 16. $e = (A^*E)_e$.

Proof. By Lemma 15 and the monotonicity of χ,

$$e = (\chi(e)E)_e \leq (\chi((\sum_{a\in\Sigma} a)^*)E)_e = ((\sum_{a\in\Sigma} \chi(a))^*E)_e = (A^*E)_e.$$

For the reverse inequality, Theorem 1 says that the identity map $e \mapsto e$ is a solution to (13), and as noted in §2.6, A^*E is the least solution of a LKA. □

Theorem 2 (Completeness). *If $d \approx e$ then $d = e$.*

Proof. Immediate from Lemmas 14 and 16. □

An interesting consequence of Lemma 16 is that the canonical solution in K_Σ is not only the least, but actually the *unique* solution. For further details, we refer the reader to [10].

5 Discussion

In this paper, we have given a new, significantly shorter proof of the completeness of the left-handed star rule of Kleene algebra.

We have shown that the left-handed star rule is needed only to guarantee the existence of least solutions. It would be interesting to explore how one could prove the existence of least solutions just using the equations assumed by Krob [12], which are of the form $M^* = \sum_{m\in M} \varepsilon_M^{-1}(m)$, for M a finite monoid.

[3] The finiteness of the subcoalgebra generated by $e \in K_\Sigma$ only requires the axioms for associativity, commutativity, and idempotency of $+$ (hence, only equations).

A well-known algorithm to obtain the minimal deterministic automaton is the *Brzozowski algorithm* [2]. Starting from a possibly nondeterministic automaton, (i) reverse the transitions, exchanging final and initial states, then (ii) perform the subset construction, removing inaccessible states; then repeat (i) and (ii). The resulting automaton is a minimal automaton for the original language.

Starting from a finite automaton (S, D, E) with a start state s, we can build an automaton $(2^S, \hat{D}, \hat{E})$ with start state E, and $\hat{D}(f) = D \circ f$, $\hat{E} = \xi(s)$, where $\xi(s)$ denotes the characteristic function of the singleton set containing s. This new automaton recognizes the reverse of the original language. Interestingly, this is also reflected in the construction of the expressions V_f for the new automaton. There is apparently a relationship to the Brzozowski construction, but the exact relationship remains to be explored.

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