# Towards a semantics for infinitary equational hybrid logic

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**Abstract**

This work-in-progress paper reports some introductory steps towards a theory of infinitary equational hybrid logic. This logic seems appropriate to express properties of reconfigurable agent systems that behave differently in different modes of operation [9]. Its semantics is obtained by endowing worlds in standard Kripke frames with algebras, each of them modelling a local configuration. The paper introduces a number of preliminary results on this semantics, including a discussion of a suitable notion of bisimulation by generalizing standard invariance results to this broad setting.

*Keywords:* Hybrid logic, Kripke semantics, infinitary logics.

## 1 Introduction

Classically, there are two main paradigms to formally capture requirements for reconfigurable agent systems: one emphasizes *behaviour* and its evolution; the other focus on *data* and their transformations. In the former systems are specified through (some variant of) *state-machines* and their evolution is expressed in terms of event occurrences and their impact in internal state configurations. In the latter, data-oriented approach the system's functionality is given in terms of input-output relations modelling operations on *data*. A specification is presented as a theory in a suitable logic, expressed over a signature which captures its syntactic interface. Its semantics is a class of concrete algebras acting as models for the specified theory.

The authors's recent work [9] aims at putting together these two approaches to pave the way to a specification methodology for reconfigurable systems. Starting from a classical state-machine specification, states are interpreted as

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different *modes* of operation and each of them is equipped with an algebra (over the system's interface) of the corresponding functionality. Technically, specifications become *structured* state-machines, states denoting *algebras*, rather than *sets*. The envisaged methodology raises a number of technical issues which this paper intends to address.

First of all there is a need for an expressive logic able to deal both with transitional behaviour and data specification. Clearly, this should be a modal language with the ability to refer to individual states, each of which stands for a local configuration. *Hybrid* logic [2,3] is thus an obvious choice. For the data part, however, *equational* logic is widely accepted as a solid, mature specification language. Actually, despite their simplicity, equations are enough to characterise all computable data structures (cf, [1]) and to describe the semantics programming languages. Moreoever, models for equational logic are (universal) algebras, well known and semantically rich structures (cf, for example, Birkhoff's characterisation of varieties [7]). On the other hand, the equational calculus is complete, and rewriting algorithms provide effective tool support for equational reasoning (as in [4]). We also consider formulas with (possible) infinite disjunctions and conjunctions. The move to an *infinitary* language [5,10] provides a suitable way to specify both liveness properties and fairness assumptions most relevant in the presence of concurrency and non determinism inherent to the kind of systems we want to capture.

The paper contributions are, thus, twofold. First an *infinitary equational hybrid* logic is introduced and its semantics given in terms of Kripke frames whose states are endowed with algebras, each of them modelling a local configuration of a reconfigurable system. Then a notion of *bisimulation* between these structures is proposed which provides a suitable notion of behavioural equivalence for comparing them. A number of preservation results studied for the hybrid propositional case (see e.g. [6]) are extended to this richer setting.

## 2 Infinitary hybrid equational logic

This section introduces the specification logic motivated in section 1. We will adopt the standard notions of equational logic and hybrid propositional logic. For a detailed exposition, the reader is referred to any standard text on each subject, for example [7] and [2], respectively.

Not only the choice for infinitary formulas [8] and equations, distinguishes this approach with respect to similar works, for example [3]. Other aspects, motivated from the specification practice, are also introduced. Such is the case, for example, of the use made of functions, which are a standard tool in algebraic specification. On the other hand, we do not consider any rigid component in the logic, allowing ample freedom in specifications.

An *equational modal similarity type* $\tau$ is a triple $\langle \Sigma, \Lambda, \mathrm{Nom} \rangle$ where $\Sigma$ is an algebraic signature and $\Lambda$ and Nom are, as above, the sets of modalities and nominals. A countable infinite set $X$ of variables is fixed. The set of $\Sigma$-terms over $X$ is defined in the usual way. The set $\mathrm{Fm}(\tau, X)$ of modal infinitary equational formulas is defined recursively as follows

(i) if $t, t'$ are $\Sigma$-terms then $t \approx t'$ is a formula;

(ii) if $\varphi$ is a formula, $\lambda \in \Lambda$, then $\neg\varphi$ and $[\lambda]\varphi$ are formulas;

(iii) if $\Gamma$ is a countable set of formulas then $\bigwedge \Gamma$ and $\bigvee \Gamma$ are formulas.

Adding nominals through the following two clauses leads to the set $\mathrm{Fm}_H(\tau, X)$

(i) the nominals are formulas;

(ii) if $\varphi$ is a formula and $i$ is a nominal, then $@_i\varphi$ is a formula.

In $\mathrm{Fm}_H(\tau, X)$ formulas defined by nominals or equations are called *atomic.* Note that each $\lambda \in \Lambda$ labels a modal box operator and each nominal $i \in \mathrm{Nom}$ is used to construct a satisfaction operator $@_i$.

**Definition 2.1** [Algebraic Kripke frame] Let $\tau$ be an equational modal similarity type. An *algebraic Kripke $\tau$-frame* is a structure $\mathcal{F} = (W, (R_\lambda)_{\lambda\in\Lambda}, (A_w)_{w\in W})$, where $W$ is a non empty set, for each $\lambda \in \Lambda$, $R_\lambda$ is a binary relation over $W$ and $A_w$ is a $\Sigma$-algebra for each $w \in W$. The relations $R_\lambda \subseteq W^2$ are called *transition relations* in $\mathcal{F}$; the elements in $W$ are usually called *possible words*, (alternatively, *states* or *modes*). The family of $\Sigma$-algebras indexed by $W$ is called the *space of configurations.*

A *pointed algebraic Kripke frame* is a pair $\langle \mathcal{F}, w\rangle$ with $w \in W$.

**Definition 2.2** [Algebraic hybrid structure] Let $\tau$ be an equational modal similarity type. An *algebraic hybrid structure* over a $\tau$-frame $\mathcal{F} = (W, (R_\lambda)_{\lambda\in\Lambda}, (A_w)_{w\in W})$ is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $V : \mathrm{Nom} \to W$ is an evaluation. For $i \in \mathrm{Nom}$, $w = V(i)$ means that state $w$ is named by $i$. $W$ is called the domain of $\mathcal{F}$. A *pointed algebraic hybrid structure* is a pair $\langle \mathcal{M}, w\rangle$, where $w \in W$.

**Definition 2.3** [Satisfaction] Let $\tau$ be an equational modal similarity type. The *satisfaction relation* $\models \subseteq W \times \mathrm{Fm}_H(\tau, X)$ on the algebraic hybrid structure $\mathcal{M} = (W, (R_\lambda)_{\lambda\in\Lambda}, (A_w)_{w\in W}, V)$ is recursively defined as follows:

(i) $\mathcal{M}, w \models i$ if $V(i) = w$;

(ii) $\mathcal{M}, w \models t \approx t'$ if $A_w \models t \approx t'$;

(iii) $\mathcal{M}, w \models @_i\varphi$ if $\mathcal{M}, s \models \varphi$, where $V(i) = s$;

(iv) $\mathcal{M}, w \models \neg\varphi$ if not $\mathcal{M}, w \models \varphi$;

(v) $\mathcal{M}, w \models \bigvee \Gamma$ if, $\mathcal{M}, w \models \varphi$ for some $\varphi \in \Gamma$;

(vi) $\mathcal{M}, w \models \bigwedge \Gamma$ if, $\mathcal{M}, w \models \varphi$ for every $\varphi \in \Gamma$;

(vii) $\mathcal{M}, w \models [\lambda]\varphi$ if, for all $w' \in W$ such that $wR_\lambda w'$ we have $\mathcal{M}, w' \models \varphi$.

If $\mathcal{M}, w \models \varphi$, we say *$\varphi$ is true at state $w$ in $\mathcal{M}$*. When $\varphi$ is satisfied at every state of $\mathcal{M}$, $\varphi$ is *valid in* $\mathcal{M}$ and we write $\mathcal{M} \models \varphi$. Finally, $\varphi$ is *valid* if $\mathcal{M} \models \varphi$ for every structure $\mathcal{M}$.

## 3 Relating models

This section introduces bisimulation for algebraic Kripke structures.

**Definition 3.1** [Bisimulation] Let $\tau$ be an equational modal similarity type. Let $\mathcal{M} = (W, (R_\lambda)_{\lambda\in\Lambda}, (A_w)_{w\in W}, V)$ and $\mathcal{M}' = (W', (R'_\lambda)_{\lambda\in\Lambda}, (A'_w)_{w\in W'}, V')$ be two algebraic hybrid structures. A *bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ is a nonempty relation $\rho \subseteq W \times W'$ such that for every pair $(w, w') \in \rho$ we have:

- Atomic conditions:
  · $\forall i \in \mathrm{Nom}$, $V(i) = w$ iff $V'(i) = w'$.
  · $\mathcal{V}(A_w) = \mathcal{V}(A'_{w'})$, i.e., $A_w$ and $A'_{w'}$ generate the same variety.
  · All points named by nominals are related by $\rho$,
- For any $\lambda \in \Lambda$, if $wR_\lambda u$ for some $u \in W$, then there is some $u' \in W'$ such that $w'R'_\lambda u'$ and $u\rho u'$ **(Zig)**,
- Similarly, in the opposite direction: for any $\lambda \in \Lambda$, if $w'R'_\lambda u'$ for some $u' \in W'$, then there is some $u \in W$ such that $wR_\lambda u$ and $u\rho u'$ **(Zag)**.

It is well known that modal satisfaction is invariant under bisimulation. The following theorem establishes a corresponding result for inifinitary equational hybrid logic. As usual, the proof proceeds by induction on the structure of formulas.

**Theorem 3.2** *The infinitary equational hybrid logic is invariant under bisimulation: let $\tau$ be an equational modal similarity type and $\rho$ a bisimulation between the $\tau$-models $\mathcal{M}$ and $\mathcal{M}'$. Then, if $w\rho w'$ we have that, for any $\varphi \in \mathrm{Fm}_H(\tau, X)$*

$$(\mathcal{M}, w) \models \varphi \text{ iff } (\mathcal{M}', w') \models \varphi.$$

The converse of this result does not hold in general: given two states of two $\tau$-models, modal equivalence between the corresponding pointed Kripke models is not in general a bisimulation. Such is the case, however, of image-countable Kripke models, as shown below. More precisely, if $\tau$ is an equational modal similarity type and $\mathcal{M}$ a $\tau$-model, we say that a $\mathcal{M}$ is *image-countable* if for each state $w \in W$ and each relation $R_\lambda$, $\lambda \in \Lambda$, the set of $\{w' \in W : wRw'\}$ is countable. Note that there is no condition about the number of relations we may have or even about the cardinality of $W$.

**Theorem 3.3** *Let $\tau$ be an equational modal similarity type with at least one constant $c$ in $\Sigma$. Let $\mathcal{M}$ and $\mathcal{M}'$ be two* image-countable *$\tau$-models. Then, for every $w \in W$ and $w' \in W'$, the following conditions are equivalent:*

(i) *$(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are bisimilar;*

(ii) *for any $\varphi \in \mathrm{Fm}_H(\tau, X)$, $(\mathcal{M}, w) \models \varphi$ iff $(\mathcal{M}', w') \models \varphi$.*

**Proof.** Suppose that for any $\varphi \in \mathrm{Fm}_H(\tau, X)$, $(\mathcal{M}, w) \models \varphi$ iff $(\mathcal{M}', w') \models \varphi$.

Let $\rho := \big\{(w, w') \in W \times W' :$ for any $\varphi \in \mathrm{Fm}_H(\tau, X), (\mathcal{M}, w) \models \varphi \;\Leftrightarrow\; (\mathcal{M}', w') \models \varphi\big\}$. The atomic conditions trivially hold. For the (Zig) condition, let $\lambda \in \Lambda$. Assume that $w\rho w'$ and let $u \in W$ such that $wR_\lambda u$. To obtain a contradiction, suppose that there is no $u' \in W'$ with $w'R_\lambda u'$ and $u\rho u'$. As in the standard case, from the condition of image-countable, the set $S' := \{u' : w'R_\lambda u'\}$ is countable. Moreover, $S'$ cannot be empty since in such case $(\mathcal{M}', w') \models [\lambda]\neg c \approx c$, which is incompatible with the fact that $(\mathcal{M}, w) \models \langle\lambda\rangle c \approx c$ ( this holds since $wR_\lambda u$). By assumption, for every $v \in S'$ there is a formula $\psi_v$ such that $(\mathcal{M}, w') \models \psi_v$ and it is false that $(\mathcal{M}', v) \models \psi_v$ (it can be at reverse order but in such case we take the negation of the formula). Consider now the conjunction $\psi = \bigwedge_{v \in S'} \psi_v$ of all of these formulas. Then, $(\mathcal{M}, w) \models \langle\lambda\rangle\psi$, and for all $v \in S'$ it is false that $(\mathcal{M}, v) \models \langle\lambda\rangle\psi$. This contradicts the fact that $w\rho w'$. The (Zag) condition can be shown in a similar way. $\square$

A consequence of the previous theorem is that any two algebraic hybrid structures with a countable set of states having the same theory are bisimilar.

## Conclusion

We briefly presented an extension of classical (propositional) hybrid logic with equations (over an algebraic signature $\Sigma$) and infinitary formulae. Some steps on its model theory were made enriching (propositional) hybrid models by endowing each state with a particular $\Sigma$-algebra supporting additional structure. A notion of bisimulation for these structures, as well as a modal equivalence theorem, was put forward. As discussed elsewhere [9], this logic, and its variants, may become an interesting alternative for specifying *reconfigurable* software systems.

In spite of the preliminary character of this work, we believe it paves the way for a number of new, relevant questions. One of them concerns the study of a complete calculus for the logic with respect to this semantics. Naturally, it should be some combination of equational logic with hybrid logic. A somehow deeper question is the study of decidability, which is crucial to the development of efficient algorithms to check properties of software specifications and their reconfigurations as captured in algebraic Kripke frames.

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