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A Hilbert-Style Axiomatisation for Equational Hybrid Logic

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Abstract This paper introduces an axiomatisation for equational hybrid logic based on previous axiomatizations and natural deduction systems for propositional and first-order hybrid logic. Its soundness and completeness is discussed. This work is part of a broader research project on the development a general proof calculus for hybrid logics.

Keywords Equational hybrid logic · Hilbert axiomatisation · Completeness

1 Introduction

Modal logics have been successfully used as specification languages for state transition systems, which, on their turn, are basic models of computational phenomena. From a proof-theoretic point of view, such logics have interesting algorithmic propri-

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eties, and, moreover, they have a natural translation to first order logic. In Computer Science, modal logics (Blackburn et al. 2001) became a popular formalism to specify reactive systems as dynamic processes which evolve in response to events. In the usual non-hybrid formulation, such logics do not allow explicit references to specific states of the underlying transition system. Such an ability, however, is considered, in a number of cases, a desirable feature of a specification formalism. Otherwise it is not possible to assert state equality or to express (local) properties of a particular state or of a group thereof.

Hybrid logic (Blackburn 2000; Indrzejczak 2007), on the other hand, *internalising* references to states as propositions, goes a step further in expressive power and provides mechanisms to handle the questions above. It proceeds by adding a new kind of symbols, called nominals, which allow referencing states (i.e., worlds in the underlying Kripke frame) as propositions. Each nominal being true at exactly one world, one can say that the world w is named by (the nominal) i if i is true at w. Besides nominals, the basic hybrid logic introduces an operator, @, that for a nominal i and a formula φ, yields a new formula $@_i\varphi$, which is true exactly when φ holds at the state named by i. This operator can also be applied to *terms*. Actually, terms can have different interpretations at different worlds and, sometimes, there is a need to restrict the interpretation of a term to a specific state. For example, we can express equality between the state named by i and the one named by j with $@_i j$, or to state the latter is accessible from the former via $@_i \Diamond j$.

In Computer Science, hybrid logics are expressive enough to model behavioural requirements of complex reactive systems. Our own previous work along this line of research is documented in Madeira et al. (2011) (see also Martins et al. 2012). The basic idea is to model systems' configurations in a suitable formalism, for example, equational logic, to express data and functional properties, and resort to hybrid logic to reason about change of configurations in response to varying context conditions. States in the underlying Kripke frame become highly structured, as they have to stand as full specifications of the system's functionality. Thus, each of them provides a local view of the system, i.e., a possible configuration. Formulas with modalities, on the other hand, express a global, dynamic view of the system's evolution. Finally, nominals allow for unambiguous reference to specific configurations. To put it in a concise way, hybrid formulas become the formal counterpart of *reconfiguration scripts*, so popular in Software Engineering but often presented in a vague, informal way.

In this context, the contribution of this paper is essentially technical: it does not add to the specification method outlined above, but brings in a fundamental ingredient to make it relevant from the point of view of applications. The paper introduces an axiomatisation for equational hybrid logic and proves its soundness and completeness. The language includes, besides variables, constants and function symbols, in contrast with more standard approaches, such as Fitting and Mendelsohn (1998) and Braüner (2005), which consider variables and predicate symbols. Another distinguishing feature of our approach is that constants are taken as non rigid terms, and can therefore be regarded as 0-ary function symbols.

Able to capture properties of both the static (equational) and dynamic (hybrid) aspects of specifications, this logic is a *lingua franca* for the method. A complete axiomatisation was therefore in order.

The paper's contribution adds to previous work on axiomatisations of propositional hybrid logic and quantified hybrid logic, as in, for example, Blackburn and Cate (2006). In particular, the technique used here to prove completeness of this extension by function symbols of quantified hybrid logic is a Henkin-style proof that closely follows the used in Blackburn and Cate (2006). Tableaux and natural deduction systems were already considered for these cases in Braüner (2005), as well as for intuitionistic hybrid logic, in Braüner and Paiva (2006).

Finally it should be remarked that this work is part of a broader research programme on the development a *general proof calculus for hybrid institutions* on top of the calculus equipping a base institution. The corresponding framework is set on a characterisation of what we have called in Martins et al. (2011) the *hybridization* process, aiming at systematically introducing nominals and hybrid quantifiers on top of popular logics.

Note that, when comparing the calculus for hybrid propositional logic with the one for hybrid first-order logic presented in Braüner (2005), a common structure pops up: both "share" rules involving sentences with nominals and satisfaction operators (i.e., formulas of an "hybrid nature") and have specific rules to reason about "atomic sentences" that come from the base institution. We intend to make explicit such a structure. The present paper is an initial step in this path, focused on the *equational* case.

The paper is structured as follows. Equational hybrid logic, its syntax and semantics, is introduced in the following section. A corresponding axiomatisation is proposed in Sect. 3. Finally, Sect. 4 proves it sound and complete. Section 5 concludes and points out a few directions for future wok.

2 Equational Hybrid Logic

Let us briefly recall, Grätzer (1979), the basic ingredients of equational logic and fix the corresponding notation.

Definition 1 (*Signature and algebra*) A *signature* Σ is a family $(\Sigma_n)_{n\in\mathbb{N}}$, where Σ_n is a set of operations symbols of arity n. Given a signature Σ, a Σ-*algebra* A is a nonempty set $|A|$ together with, for each $f \in \Sigma_n$ a function $A_f : |A| \times \cdots \times |A| \to |A|$. An homomorphism between two algebras A and A' consists of a map $h : |A| \to |A'|$ such that, for any $f \in \Sigma_n$ and any $a_1, \ldots, a_n \in |A|$, $h(A_f(a_1, \ldots, a_n)) = A'_f(h(a_1), \ldots, h(a_n))$.

Let X denote a set of *variables*. Then, we define

Definition 2 (*Terms and equations*) The set of Σ-*terms* (over X), $T(\Sigma, X)$, is recursively defined by

- for any $x \in X$, $x \in T(\Sigma, X)$;
- for any $f \in \Sigma_n$, and all term $t_i \in T(\Sigma, X)$, $i = 1, \ldots, n$, $f(t_1, \ldots, t_n) \in T(\Sigma, X)$.

A Σ-*equation* in the variables X is an expression $t \approx t'$, for $t, t' \in T(\Sigma, X)$. The set of all Σ-equations in the variables X is represented by $\mathrm{Eq}_\Sigma(\mathrm{X})$ or, simply by Eq_Σ when

X is clear from the context. Finally, an equation $t \approx t'$ is satisfied by an algebra A, in symbols $A \models t \approx t'$, if for any assignment $s : X \to |A|$, we have $\bar{s}(t) \approx \bar{s}(t')$, where $\bar{s}$ is the unique homomorphism extension of $s : X \to |A|$ to $T(\Sigma, X)$. Whenever A is the term algebra s is called a *substitution*.

We may now introduce the syntax and a semantics for *equational hybrid logic*, $\mathcal{EQL}(@)$. As expected, its interpretation is based on a suitable generalisation of Kripke models.

Definition 3 An *equational hybrid similarity type* τ is a triple $\langle \Sigma, X, \mathrm{NOM} \rangle$ where Σ is an algebraic signature, X is a countable infinite set of variables and NOM is a set of symbols, called *nominals*. The set $\mathrm{Term}(\tau)$ of *hybrid Σ-terms over X*, abbreviated to *terms* in the sequel, is recursively defined by

- for any $x \in X$, $x \in \mathrm{Term}(\tau)$;
- for any $c \in \Sigma_0$, $c \in \mathrm{Term}(\tau)$;
- for any $f \in \Sigma_n$, and all terms $t_1, \ldots, t_n \in \mathrm{Term}(\tau)$, $f(t_1, \ldots, t_n) \in \mathrm{Term}(\tau)$;
- for any $t \in \mathrm{Term}(\tau)$ and $i \in \mathrm{NOM}$, $@_i t \in \mathrm{Term}(\tau)$.

Two different sorts of terms are to be distinguished: the standard terms, i.e. elements of $T(\Sigma, X)$, called *basic terms*, and, on the other hand, those terms whose outmost operator is @, known as *rigidified terms*. The set $@T(\Sigma, X)$ of rigidified terms can also be recursively defined by

- for any $x \in X$ and $i \in \mathrm{NOM}$, $x, @_i x \in @T(\Sigma, X)$;
- for any $c \in \Sigma_0$ and $i \in \mathrm{NOM}$, $@_i c \in @T(\Sigma, X)$;
- for any $f \in \Sigma_n$, $i \in \mathrm{NOM}$ and all terms $t_1, \ldots, t_n \in \mathrm{Term}(\tau)$, $@_i f(t_1, \ldots, t_n) \in @T(\Sigma, X)$.

The equational hybrid language $\mathrm{Fm}(\tau)$, includes a modality $\Diamond$ and a *reference* operator, $@_i$, for each nominal $i \in \mathrm{NOM}$.

Definition 4 The set of $\mathrm{Fm}(\tau)$ of equational hybrid formulas is defined recursively as follows

- all nominals are formulas;
- if t, t' are Σ-terms then $t \approx t'$ is a formula;
- if φ is a formula and i is a nominal, then $@_i\varphi$ is a formula;
- if φ is a formula, then $\neg\varphi$ and $\Diamond\varphi$ are formulas;
- if φ and ψ are formulas then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are formulas.

Nominals and equations are called *atomic* formulas.

As usual, the following abbreviations are considered: $\Box\varphi := \neg\Diamond\neg\varphi$, $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. Moreover, for any natural $n > 0$, $\Box^n\gamma$ is recursively defined by $\Box^0\gamma := \gamma$ and $\Box^{n+1}\gamma := \Box(\Box^n\gamma)$ for $n > 0$. Finally, for Γ finite, we write $\bigwedge \Gamma$ to denote the finite conjunction of all formulas in Γ.

Definition 5 (*Algebraic Kripke frame*) Let $\tau = \langle \Sigma, X, \mathrm{NOM} \rangle$ be an equational hybrid similarity type. An *algebraic Kripke τ-frame* is a structure $\mathcal{F} = (W, R, (A_w)_{w \in W})$, where W is a non empty set, $R \subseteq W^2$ is a binary relation over W and, for all $w \in W$, each A_w is a Σ-algebra such that all algebras $(A_w)_{w \in W}$ share the same carrier $|A|$ (i.e., $\mathcal{F}$ has *constant domains*). A *pointed algebraic Kripke frame* is a pair $\langle \mathcal{F}, w \rangle$ with $w \in W$. The class of all algebraic Kripke frames over τ is denoted by $\mathrm{AlgK}(\tau)$.

As in modal logic, relation R is called the *accessibility* or *transition* relation, whereas elements of W are known as *possible words*, *states* or *modes*. The family of Σ-algebras indexed by W forms the *space of configurations* for $\mathcal{F}$.

Definition 6 (*Algebraic hybrid structure*) Let τ be an equational hybrid similarity type. An *algebraic hybrid structure* over τ-frame $\mathcal{F} = (W, R, (A_w)_{w \in W})$ is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $V : \mathrm{NOM} \to W$ is an evaluation, giving, for each $i \in \mathrm{NOM}$, the state $V(i)$ it refers to. W is called the domain of $\mathcal{F}$. A *pointed algebraic hybrid structure* is a pair $\langle \mathcal{M}, w \rangle$, where $\mathcal{M}$ is an algebraic hybrid structure and $w \in W$. The class of all algebraic hybrid structures over τ is denoted by $\mathrm{AlgH}(\tau)$.

Sub-structures are defined as usual from a set of states closed for transitions. Formally,

Definition 7 (*Sub-structure*) $\mathcal{M}$ is *an algebraic hybrid sub-structure of* $\mathcal{M}'$, in symbols $\mathcal{M} \preceq \mathcal{M}'$, if, $W \subseteq W'$, $R = R' \cap W^2$, $V(i) = V'(i)$ for any $i \in \mathrm{NOM}$ and, $A_w = A'_w$ for any $w \in W$. Moreover, given $W_0 \subseteq W$, the *algebraic hybrid structure generated by* W_0 *in* $\mathcal{M}$, denoted by $\mathcal{M}_{W_0}$, is the smallest (i.e., with the smallest set of states) algebraic hybrid sub-structure $\mathcal{M}$ of $\mathcal{M}'$, such that $W_0 \subseteq W$.

The interpretation of operation symbols in a particular algebra A_w, for a state w, is represented as $f_{A_w} : A_w^n \to A_w$ ($c_{A_w} : A_w$ for constants). Note that both constant and function symbols are interpreted non-rigidly, i.e., they may have different values in different worlds. Variables, however, are rigid, i.e., their evaluation is the same in every world. The interpretation of terms and formulas in a given algebraic hybrid structure is, thus, as follows:

Definition 8 Let $\mathcal{M}$ be an algebraic hybrid structure and $g : X \to |A|$ an assignment. The interpretation of terms is recursively defined as follows:

- if $t \in X$, $[t]^{\mathcal{M},w,g} = g(t)$;
- if $t \in \Sigma_0$, $[t]^{\mathcal{M},w,g} = V_w(t)$;
- if $t_1, \ldots, t_n \mathrm{Term}(\tau)$, $f \in \Sigma_n$;
  $[f(t_1, \ldots, t_n)]^{\mathcal{M},w,g} = V_w(f)([t_1]^{\mathcal{M},w,g}, \ldots, [t_n]^{\mathcal{M},w,g})$;
- if $t \in \mathrm{Term}(\tau)$, $i \in \mathrm{NOM}$, $[@_i t]^{\mathcal{M},w,g} = [t]^{\mathcal{M},V(i),g}$.

At each world $w \in W$, satisfaction of formulas is given by

$$\begin{aligned}
\mathcal{M}, w \models i & \quad \text{if } V(i) = w \\
\mathcal{M}, w \models t_1 \approx t_2 & \quad \text{if, } [t_1]^{\mathcal{M},w,g} = [t_2]^{\mathcal{M},w,g} \text{ for each } g : X \to A \\
\mathcal{M}, w \models @_i \varphi & \quad \text{if } \mathcal{M}, v \models \varphi, \text{ where } V(i) = v \\
\mathcal{M}, w \models \neg \varphi & \quad \text{if } \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \quad \text{if } \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Diamond \varphi & \quad \text{if there is a } v \text{ such that } wRv \text{ and } \mathcal{M}, v \models \varphi
\end{aligned}$$

A formula φ is said to be *true* in world w if and only if $\mathcal{M}, w \models \varphi$. It is *valid for a structure* $\mathcal{M}$, denoted by $\mathcal{M} \models \varphi$, if and only if, for every world w, $\mathcal{M}, w \models \varphi$. Finally, a formula φ is *valid* ($\models \varphi$) if and only if, for every $\mathcal{M}$ and every world w, $\mathcal{M}$, $w \models \varphi$.

These definitions extend to sets of formulas in the usual way; for example $\mathcal{M} \models \Gamma$ if $\mathcal{M} \models \varphi$ for any $\varphi \in \Gamma$.

Lemma 1 *Let τ be an equational hybrid similarity type, $\mathcal{M}$ an algebraic hybrid structure, $W_0 \subseteq W$, $w \in W_0$ and $\varphi \in \mathrm{Fm}(\tau)$. Then*

$$\mathcal{M}_{W_0}, w \models \varphi \quad \Leftrightarrow \quad \mathcal{M}, w \models \varphi.$$

Proof The left to right implication is proved by induction, the basic observation for the modal case being the fact that a sub-structure is closed for transitions. The other direction is immediate from definitions.

Two different notions of semantic consequence, a *local* and a *global* one, are defined below. Note that, since our focus is the class of all algebraic hybrid structures, they are just given for this case. However, both definitions extend naturally to sub-classes as in standard modal logic.

Definition 9 Let τ be an equational hybrid similarity type and $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\tau)$. Then

- Γ is *satisfiable* if there is $\mathcal{M} \in \mathrm{AlgH}(\tau)$ such that $\mathcal{M} \models \Gamma$;
- $\Gamma \models \varphi$ if, for all $\mathcal{M} \in \mathrm{AlgH}(\tau)$ and for all $w \in W$, $\mathcal{M}, w \models \Gamma$ implies $\mathcal{M}, w \models \varphi$;
- $\Gamma \Vdash\!\models \varphi$ if, for all $\mathcal{M} \in \mathrm{AlgH}(\tau)$ $\mathcal{M} \models \Gamma$ implies $\mathcal{M} \models \varphi$.

Relations $\models$ and $\Vdash\!\models$ are called, respectively, the *local* and *global consequence* on $\mathrm{AlgH}(\tau)$.

Definition 10 An *inference rule* is a pair $\langle \Gamma, \varphi \rangle$, typically written as $\dfrac{\Gamma}{\varphi}$, where Γ is a finite set of formulas and φ a formula. A rule is *valid*, if $\models \Gamma$ implies $\models \varphi$, and *normal* if $\Gamma \models \varphi$.

Proposition 1 *A rule $\dfrac{\Gamma}{\varphi}$ is normal iff $\models \bigwedge \Gamma \to \varphi$, where $\bigwedge \Gamma$ denotes the finite conjunction of all formulas in Γ.*

Proof

$$\begin{aligned}
&\Gamma \models \varphi \\
&\quad \Leftrightarrow \text{for all, } \mathcal{M} \in \mathrm{AlgH}(\tau) \text{ and for all } w \in W, \mathcal{M}, w \models \Gamma \text{ implies } \mathcal{M}, w \models \varphi \\
&\quad \Leftrightarrow \text{for all } \mathcal{M} \in \mathrm{AlgH}(\tau) \text{ and for all } w \in W, \mathcal{M}, w \models \bigwedge \Gamma \text{ implies } \mathcal{M}, w \models \varphi \\
&\quad \Leftrightarrow \text{for all } \mathcal{M} \in \mathrm{AlgH}(\tau) \text{ and for all } w \in W, \mathcal{M}, w \models \bigwedge \Gamma \to \varphi \\
&\quad \Leftrightarrow \models \bigwedge \Gamma \to \varphi.
\end{aligned}$$

These definitions extend naturally to classes of algebraic hybrid structures. Note also that, as usual, explicit reference to Γ is omitted whenever Γ is the empty set.

As in standard modal logic the following result holds,

Lemma 2 *Let τ be an equational modal similarity type and $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\tau)$. Then*

(1) $\Gamma \models \varphi \Rightarrow \Gamma \Vdash\!\models \varphi$;
(2) $\Gamma \Vdash\!\models \varphi \Leftrightarrow \bigcup_{n \geq 0} \Box^n \Gamma \models \varphi$, where $\Box^n \Gamma = \{\Box^n \gamma : \gamma \in \Gamma\}$;
(3) $\Gamma \models \varphi \Rightarrow \Gamma \cup \{\neg\varphi\}$ is not satisfiable.

Proof (1) Let $\mathcal{M} \in \mathrm{AlgH}(\tau)$ such that $\mathcal{M} \models \Gamma$. Clearly, for every $w \in W$, $\mathcal{M}, w \models \Gamma$ and hence, by hypothesis, $\mathcal{M}, w \models \varphi$. Therefore, $\mathcal{M} \models \varphi$, which establishes $\Gamma \Vdash\!\models \varphi$.
(2) Suppose $\Gamma \Vdash\!\models \varphi$. Let $\mathcal{M} \in \mathrm{AlgH}(\tau)$ and $w \in W$ such that $\mathcal{M}, w \models \bigcup_{n \geq 0} \Box^n \Gamma$. Now note that $\mathcal{M}_{\{w\}} \models \Gamma$ because

$$\begin{aligned} &\mathcal{M}, w \models \bigcup_{n\geq 0} \Box^n \Gamma \\ &\quad \Leftrightarrow \mathcal{M}, w \models \Box\,\Gamma \ \wedge\ \mathcal{M}, w \models \bigcup_{n>0} \Box^n \Gamma \\ &\quad \Leftrightarrow \mathcal{M}, w \models \Box\,\Gamma \ \wedge\ \mathcal{M}, w \models \Box \left\{ \bigcup_{n\geq 0} \Box^n \Gamma \right\} \\ &\quad \Leftrightarrow \mathcal{M}, w \models \Box\,\Gamma \ \wedge\ \forall_{w' \in W}\ wRw' \Rightarrow \mathcal{M}, w' \models \bigcup_{n\geq 0} \Box^n \Gamma. \end{aligned}$$

Therefore, $\mathcal{M}, w' \models \Gamma$. Iterating this calculation shows that for every world z accessible, in one or more steps, from w, $\mathcal{M}, z \models \Gamma$. By Definition 7, these are exactly the worlds of $\mathcal{M}_{\{w\}}$, which establishes $\mathcal{M}_{\{w\}} \models \Gamma$.

Hence, by hypothesis, $\mathcal{M}_{\{w\}} \models \varphi$. Thus $\mathcal{M}_{\{w\}}, w \models \varphi$. By Lemma 1, $\mathcal{M}, w \models \varphi$. The reciprocal implication holds from (1).
(3)

$$\begin{aligned} &\Gamma \cup \{\neg\varphi\} \ \text{ is satisfiable} \\ &\quad \Leftrightarrow \text{ there is } \mathcal{M} \in \mathrm{AlgH}(\tau) \text{ such that for all } w \in W, \mathcal{M}, w \models \Gamma \cup \{\neg\varphi\} \\ &\quad \Leftrightarrow \text{ there is } \mathcal{M} \in \mathrm{AlgH}(\tau) \text{ such that for all } w \in W, \mathcal{M}, w \models \Gamma, \mathcal{M}, w \models \neg\varphi \\ &\quad \Rightarrow \text{ there is } \mathcal{M} \in \mathrm{AlgH}(\tau) \text{ and there is } w \in W, \mathcal{M}, w \models \Gamma, \mathcal{M}, w \models \neg\varphi \\ &\quad \Leftrightarrow \text{ it is false that } \Gamma \models \varphi. \end{aligned}$$

3 An Axiomatisation

This section proposes an axiomatisation K_τ for equational hybrid logic, given an equational hybrid similarity type τ. Let $\varphi, \psi, \xi \in \mathrm{Fm}(\tau)$ and $i, j \in \mathrm{NOM}$. Then,

Axioms

(*taut*) all instances of propositional tautologies

$(K_\Box)$ $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Axioms for @ operator and $\approx$:

$(K_@)$ $@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi)$

$(K_{@\approx})$ $@_i(t_1 \approx t_2) \leftrightarrow (@_i t_1 \approx @_i t_2), t_1, t_2 \in T(\Sigma, X)$

$(Selfdual_@)$ $@_i\varphi \leftrightarrow \neg @_i \neg\varphi$

$(Ref_@)$ $@_i i$

$(Sym_@)$ $@_i j \leftrightarrow @_j i$

(Nom) $(@_i j \wedge @_j\varphi) \to @_i\varphi$

$(Nom_\approx)$ $@_i j \to (@_i t \approx @_j t), t \in T(\Sigma, X)$

$(Agree)$ $@_i @_j\varphi \leftrightarrow @_j\varphi$

$(Agree_\approx)$ $@_i(t_1 \approx t_2) \leftrightarrow (t_1 \approx t_2), t_1, t_2 \in @T(\Sigma, X)$

$(Intro)$ $i \to (\varphi \leftrightarrow @_i\varphi)$

$(Back)$ $\Diamond @_i\varphi \to @_i\varphi$

$(Loc_\approx)$ $x \approx @_i x,\ x \in X$

Local axioms for equational logic:

$(Ref_\approx)$ $t_1 \approx t_1, t_1 \in \mathrm{Term}(\tau)$

$(EQSym)$ $@_i(t_1 \approx t_2) \to @_i(t_2 \approx t_1), t_1, t_2 \in \mathrm{Term}(\tau)$

$(EQTrans)$ $\big(@_i(t_1 \approx t_2) \wedge @_i(t_2 \approx t_3)\big) \to @_i(t_1 \approx t_3), t_1, t_2, t_3 \in \mathrm{Term}(\tau)$

$(EQFun)$ $\big(@_i(t_1 \approx t_1') \wedge \cdots \wedge @_i(t_n \approx t_n')\big) \to @_i(f(t_1, \ldots, t_n) \approx f(t_1', \ldots, t_n')), t_i, t_i' \in \mathrm{Term}(\tau)$, for $i = 1, \ldots, n$

Rules

(MP) $\dfrac{\varphi \to \psi \quad \varphi}{\psi}$

$(Gen_@)$ $\dfrac{\varphi}{@_i\varphi}$

$(Gen_\Box)$ $\dfrac{\varphi}{\Box\varphi}$

(BG) $\dfrac{@_i\Diamond j \to @_j\varphi}{@_i\Box\varphi}$, if $j \neq i$ and j does not occur in φ

$(Name)$ $\dfrac{@_i\varphi}{\varphi}$, where i does not occur in φ

$(Subs_\approx)$ $\dfrac{\varphi \to (t \approx t')}{\varphi \to \overline{s}(t \approx t')}$, where $\overline{s} : \mathrm{Term}(\tau) \longrightarrow \mathrm{Term}(\tau)$ is the canonical extension to terms of a substitution $s : X \longrightarrow \mathrm{Term}(\tau)$

$(Subs)$ $\dfrac{\varphi}{\varphi'}$, where φ' is any formula obtained from φ by replacing nominals by nominals and variables by rigidified terms.

A few comments on some of the axioms are in order. First note that axiom $(Agree_\approx)$ is valid only for rigidified terms and variables, since only these two types of terms are interpreted rigidly. Thus, their equality proved at one specific world can be generalised to all worlds.

The $(Name)$ rule deserves special attention. This rule plays an essentially technical role in the completeness proof. The rule is valid, but not normal, which partially explains why we prove just a weak soundness result, i.e., restricted to theorems, in the next section.

Unlike most orthodox approaches to modal logic where the emphasis is put on finding axiomatisations for the valid formulas, we will consider here consequence relations induced by the axioms and inference rules. In this context there are two distinct consequence relations worth to study: a *global* and a *local* one, the latter playing traditionally a more relevant role in modal logic. Formally,

Definition 11 Let τ be an equational hybrid similarity type.

- A formula φ is *derivable* from a set of formulas Γ in K_τ, represented by $\Gamma \Vdash_\tau \varphi$, if and only if there is a sequence of formulas $\varphi_0, \ldots, \varphi_{n-1}$ where $\varphi_{n-1} = \varphi$ and, for each $i \in \mathbb{N}$, $i < n$, φ_i is either an axiom, an element of Γ or obtained from formulas appearing previously in the sequence, by applying one of the inference rules. A formula φ is called a *theorem* if and only if $\Vdash_\tau \varphi$.
- A formula φ is a *local* consequence of a set of formulas Γ in K_τ, represented by $\Gamma \vdash_\tau \varphi$, if $\Vdash_\tau \varphi$ or there are $\varphi_0, \ldots, \varphi_{n-1}$ in Γ such that $\Vdash_\tau (\varphi_0 \wedge \cdots \wedge \varphi_{n-1}) \to \varphi$.

The use of τ, as a subscript to both $\Vdash$ and $\vdash$, is omitted whenever the similarity type is clear from the context. As it was the case for semantic consequence, *local* and *global* consequence can be related in the following way:

Lemma 3

$$\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \Vdash \varphi$$

Proof Suppose $\Gamma \vdash \varphi$. Then there are $\varphi_0, \ldots, \varphi_{n-1}$ in Γ such $\Vdash (\varphi_0 \wedge \cdots \wedge \varphi_{n-1}) \to \varphi$. Clearly, $\Gamma \Vdash \varphi_0 \wedge \cdots \wedge \varphi_{n-1}$. Hence, by (MP), $\Gamma \Vdash \varphi$.

It is easy to see that we have an equivalence when Γ is empty. However, the reciprocal of this result does not hold. For example, from $(Gen_\Box)$ we have $\{x \approx f(x,x)\} \Vdash \Box(x \approx f(x,x))$, however $\Vdash x \approx f(x,x) \to \Box(x \approx f(x,x))$ is not true. This shows that a deduction theorem for the global consequence relation $\Vdash$ does not exist. It does, however, locally, for $\vdash$.

Lemma 4 (DDT—Deduction detachment for $\vdash$)

$$\Gamma \cup \{\varphi\} \vdash \psi \;\Leftrightarrow\; \Gamma \vdash \varphi \to \psi.$$

Proof

$$\begin{aligned}
&\Gamma \cup \{\varphi\} \vdash \psi \\
&\quad \Leftrightarrow \exists \varphi_0, \ldots, \varphi_{n-1} \in \Gamma \text{ such that } \Vdash (\varphi_0 \wedge \cdots \wedge \varphi_{n-1} \wedge \varphi) \to \psi \\
&\quad \Leftrightarrow \exists \varphi_0, \ldots, \varphi_{n-1} \in \Gamma \text{ such that } \Vdash (\varphi_0 \wedge \cdots \wedge \varphi_{n-1}) \to (\varphi \to \psi) \\
&\quad \Leftrightarrow \Gamma \vdash \varphi \to \psi.
\end{aligned}$$

To keep the paper self contained, the next lemma lists, without proofs, some basic facts.

Lemma 5

(1) if $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$ then $\Delta \vdash \varphi$.
(2) if $\Gamma \vdash \varphi$ and $\{\varphi\} \vdash \psi$ then $\Gamma \vdash \psi$.
(3) if $\Gamma \vdash \Delta$ and $\Delta \vdash \varphi$ then $\Gamma \vdash \varphi$.
(4) if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$ then $\Gamma \vdash \psi$.

The following proposition introduces a few theorems about K_τ that will be useful on the proof of completeness for this axiomatisation.

Proposition 2

$$
\begin{array}{ll}
(K^{-1}{}_{@}) & \Vdash (@_i\varphi \rightarrow @_i\psi) \rightarrow @_i(\varphi \rightarrow \psi) \\
(\mathit{Nom}) & \Vdash @_i j \rightarrow (@_i\varphi \rightarrow @_j\varphi) \\
(\mathit{Sym}) & \Vdash @_i j \rightarrow @_j i \\
(\mathit{Bridge}) & \Vdash @_i\Diamond j \wedge @_j\varphi \rightarrow @_i\Diamond\varphi \\
(\mathit{Conj}) & \Vdash @_i(\varphi \wedge \psi) \leftrightarrow (@_i\varphi \wedge @_i\psi) \\
(\mathit{Elim}) & \Vdash (i \wedge @_i\varphi) \rightarrow \varphi \\
(\mathit{EQAgree}) & \Vdash @_i(t_1 \approx @_j t_2) \leftrightarrow (@_i t_1 \approx @_j t_2), t_1, t_2 \in T(\Sigma, X)
\end{array}
$$

Proof The proofs of $(Conj)$ and $(EQAgree)$ are presented below. See Blackburn and Cate (2006) for the remaining cases.

$(Conj)$:

$$
\begin{array}{cl}
\hline
@_i\neg(\varphi \wedge \neg\psi) \leftrightarrow \neg(@_i\varphi \wedge \neg @_i\psi) & (K_@), (K_@^{-1}) \\
\hline
\neg @_i(\varphi \wedge \neg\psi) \leftrightarrow \neg(@_i\varphi \wedge \neg @_i\psi) & (\mathit{Selfdual}_@) \\
\hline
@_i(\varphi \wedge \neg\psi) \leftrightarrow (@_i\varphi \wedge \neg @_i\psi) & \\
\hline
@_i(\varphi \wedge \neg\psi) \leftrightarrow (@_i\varphi \wedge @_i\neg\psi) & (\mathit{Selfdual}_@) \\
\hline
@_i(\varphi \wedge \psi) \leftrightarrow (@_i\varphi \wedge @_i\psi) &
\end{array}
$$

$(EQAgree)$:

$$
\dfrac{
K_{@\approx}\,\dfrac{}{@_i(t_1 \approx @_j t_2) \leftrightarrow @_i t_1 \approx @_i @_j t_2}
\qquad
\dfrac{\dfrac{\dfrac{}{x \approx @_i x}(Loc_\approx)}{@_i @_j t_2 \approx @_j t_2}(Subs)}{@_i t_1 \approx @_i @_j t_2 \leftrightarrow @_i t_1 \approx @_j t_2}
}{
@_i(t_1 \approx @_j t_2) \leftrightarrow (@_i t_1 \approx @_j t_2)
}
$$

Definition 12 A rule $\dfrac{\Gamma}{\varphi}$ is

– *(locally) derivable in* K_τ if $\Gamma \vdash \varphi$;
– *admissible in* K_τ if $\Vdash \Gamma$ implies $\Vdash \varphi$.

It is clear that derivability implies admissibility; moreover admissibility is easier to prove. The following lemma gives a useful characterisation of derivability,

Lemma 6 *The following conditions are equivalent:*

(1) $\dfrac{\Gamma}{\varphi}$ *is (locally) derivable in* K_τ*;*
(2) $\Vdash \bigwedge \Gamma \to \varphi$*;*
(3) For every set of formulas Δ, $\Delta \vdash \bigwedge \Gamma \Rightarrow \Delta \vdash \varphi$*,*

where $\bigwedge \Gamma$ *denotes the finite conjunction of all formulas in* Γ*.*

Proof By definition, $\Gamma \vdash \varphi$ iff $\Vdash \bigwedge \Gamma \to \varphi$. Suppose now (2) and let Δ be such that $\Delta \vdash \bigwedge \Gamma$. Then, by definition of $\vdash$, there is a finite $\Delta_0 \subseteq \Delta$ such that $\Vdash \bigwedge \Delta_0 \to \bigwedge \Gamma$. Then, by (2), $\Vdash \bigwedge \Delta_0 \to \varphi$, i.e., $\Delta \vdash \varphi$. Clearly, by taking $\Delta = \Gamma$ we have that (3) implies (2).

In Lemma 5 we have shown that *Modus ponens* is a derivable rule in K_τ. The following proposition presents two admissible rules that will be relevant in the sequel.

Proposition 3 *The following rules are admissible in* K_τ*:*

$$(Paste_\Diamond) \quad \frac{(@_i\Diamond j \wedge @_j\varphi) \to \psi}{@_i\Diamond\varphi \to \psi}, \quad if\ j \neq i\ does\ not\ occur\ in\ \varphi\ or\ \psi$$

$$(Name') \quad \frac{i \to \varphi}{\varphi}, \quad where\ i\ does\ not\ occur\ in\ \varphi.$$

Proof $(Paste_\Diamond)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{(@_i\Diamond j \wedge @_j\varphi) \to \psi}{(@_i\Diamond j \to \neg @_j\varphi) \vee \psi}}{(@_i\Diamond j \to @_j\neg\varphi) \vee \psi}\,(Selfdual@)}{@_i\Box\neg\varphi \vee \psi}\,(BG)}{@_i\neg\Diamond\varphi \vee \psi}}{\neg @_i\Diamond\varphi \vee \psi}\,(Selfdual@)}{@_i\Diamond\varphi \to \psi}$$

$(Name')$:

$$\cfrac{\cfrac{\cfrac{\cfrac{i \to \varphi}{@_i(i \to \varphi)}\,(Gen_@)}{@_i i \to @_i\varphi}\,(K_@)}{@_i\varphi}\,(Ref@)}{\varphi}\,(Name)$$

Corollary 1 *Let* $\Gamma \cup \{\varphi, \psi\}$ *be a set of formulas and* i, j *nominals. Then,*

(1) if $j \neq i$ *does not occur in* $\Gamma \cup \{\varphi, \psi\}$*,*

$$\Gamma \vdash (@_i\Diamond j \wedge @_j\varphi) \to \psi \;\Rightarrow\; \Gamma \vdash @_i\Diamond\varphi \to \psi.$$

(2) if i *does not occur in* $\Gamma \cup \{\varphi\}$*,*

$$\Gamma \vdash i \to \varphi \;\Rightarrow\; \Gamma \vdash \varphi.$$

Proof (1) Suppose that $\Gamma \vdash (@_i\Diamond j \wedge @_j\varphi) \rightarrow \psi$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Vdash \bigwedge \Gamma_0 \rightarrow (@_i\Diamond j \wedge @_j\varphi) \rightarrow \psi$. Then, $\Vdash (@_i\Diamond j \wedge @_j\varphi) \rightarrow (\bigwedge \Gamma_0 \rightarrow \psi)$. By admissibility we have $\Vdash @_i\Diamond\varphi \rightarrow (\bigwedge \Gamma_0 \rightarrow \psi)$. Hence, $\Vdash \bigwedge \Gamma_0 \rightarrow (@_i\Diamond\varphi \rightarrow \psi)$. i.e., by Lemma 6, $\Gamma \vdash @_i\Diamond\varphi \rightarrow \psi$.

The proof is similar for the case (2).

4 Soundness and Completeness

This section introduces the paper's main contribution: the detailed proof of both soundness and completeness of the proposed axiomatisation.

Lemma 7 *All the axioms of equational hybrid logic are valid formulas.*

Proof The proof is given for all axioms involving $\approx$. The remaining cases are standard. ($K_{@\approx}$) Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Then,

$$\begin{aligned}
&\mathcal{M}, w \vDash @_i(t_1 \approx t_2)\\
&\quad\Leftrightarrow [t_1]^{\mathcal{M},v,g} = [t_2]^{\mathcal{M},v,g} \text{ for every assignment } g \text{ and } v = V(i)\\
&\quad\Leftrightarrow [@_i t_1]^{\mathcal{M},w,g} = [@_i t_2]^{\mathcal{M},w,g} \text{ for every assignment } g\\
&\quad\Leftrightarrow \mathcal{M}, w \vDash @_i t_1 \approx @_i t_2
\end{aligned}$$

Therefore, $\mathcal{M}, w \vDash @_i(t_1 \approx t_2) \leftrightarrow @_i t_1 \approx @_i t_2$.

($Nom_\approx$) Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$ such that $\mathcal{M}, w \vDash @_i j$. Thus, $V(i) = V(j)$. Since, $[@_i t]^{\mathcal{M},w,g} = [@_j t]^{\mathcal{M},w,g}$, for every assignment g. Therefore, $\mathcal{M}, w \vDash @_i t \approx @_j t$.

($Agree_\approx$) Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Thus,

$$\begin{aligned}
&\mathcal{M}, w \vDash @_i(t_1 \approx t_2)\\
&\quad\Leftrightarrow \mathcal{M}, v \vDash t_1 \approx t_2 \text{ with } v = V(i)\\
&\quad\Leftrightarrow [t_1]^{\mathcal{M},v,g} = [t_2]^{\mathcal{M},v,g} \text{ for every assignment } g \text{ and } v = V(i)\\
&\quad\Leftrightarrow [t_1]^{\mathcal{M},w,g} = [t_2]^{\mathcal{M},w,g} \text{ for every assignment } g, \text{ since } t_1, t_2 \in @T(\Sigma, X)\\
&\quad\Leftrightarrow \mathcal{M}, w \vDash t_1 \approx t_2
\end{aligned}$$

Therefore, $\mathcal{M}, w \vDash @_i(t_1 \approx t_2) \leftrightarrow t_1 \approx t_2$.

($Loc_\approx$) Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Since $[x]^{\mathcal{M},w,g} = g(x) = [x]^{\mathcal{M},v,g} = [@_i x]^{\mathcal{M},w,g}$, for $V(i) = v$, $\mathcal{M}, w \vDash t \approx @_i t$.

($EQRef$) Trivial.

($EQSym$) Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Suppose $\mathcal{M}, w \vDash @_i(t_1 \approx t_2)$, that is, $\mathcal{M}, v \vDash t_1 \approx t_2$, where $V(i) = v$. So, $[t_1]^{\mathcal{M},v,g} = [t_2]^{\mathcal{M},v,g}$, for every assignment g. This implies that $\mathcal{M}, v \vDash t_2 \approx t_1$ and, consequently, $\mathcal{M}, w \vDash @_i(t_2 \approx t_1)$. Therefore, $\mathcal{M}, w \vDash @_i(t_1 \approx t_2) \rightarrow @_i(t_2 \approx t_1)$.

($EQTrans$) Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Suppose $\mathcal{M}, w \vDash @_i(t_1 \approx t_2)$ and $\mathcal{M}, w \vDash @_i(t_2 \approx t_3)$. Then, $\mathcal{M}, v \vDash t_1 \approx t_2$ and $\mathcal{M}, v \vDash t_2 \approx t_3$, with $V(i) = v$. So, $[t_1]^{\mathcal{M},v,g} = [t_2]^{\mathcal{M},v,g} = [t_3]^{\mathcal{M},v,g}$, for every assignment g and $V(i) = v$. This implies that $\mathcal{M}, v \vDash t_1 \approx t_3$, with $V(i) = v$. Hence, $\mathcal{M}, w \vDash @_i(t_1 \approx t_3)$. Therefore, $\mathcal{M}, w \vDash \big(@_i(t_1 \approx t_2) \wedge @_i(t_2 \approx t_3)\big) \rightarrow @_i(t_1 \approx t_3)$.

$(EQFun)$ Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Suppose $\mathcal{M}, w \models \big(@_i(t_1 \approx t_1') \wedge \cdots \wedge @_i(t_n \approx t_n')\big)$. Then, $\mathcal{M}, w \models @_i(t_k \approx t_k')$, $k = 1, \ldots, n$. That is, $\mathcal{M}, v \models t_k \approx t_k'$, $k = 1, \ldots, n$, whith $V(i) = v$. So, $[t_k]^{\mathcal{M},v,g} = [t_k']^{\mathcal{M},v,g}$, $k = 1, \ldots, n$, for every assignment g and $V(i) = v$. This implies that $[f(t_1, \ldots t_n)]^{\mathcal{M},v,g} = [f(t_1', \ldots, t_n')]^{\mathcal{M},v,g}$, for every assignment g and $V(i) = v$. Thus, $\mathcal{M}, v \models f(t_1, \ldots, t_n) \approx f(t_1', \ldots, t_n')$ and, consequently, $\mathcal{M}, w \models @_i(f(t_1, \ldots, t_n) \approx f(t_1', \ldots, t_n'))$. Therefore, $\mathcal{M}, w \models \big(@_i(t_1 \approx t_1') \wedge \cdots \wedge @_i(t_n \approx t_n')\big) \rightarrow @_i(f(t_1, \ldots, t_n) \approx f(t_1', \ldots, t_n'))$.

The following lemma establishes rule validity.

Lemma 8 *All the rules of equational hybrid logic are valid.*

Proof (MP) Suppose that $\models \varphi$ and $\models \varphi \rightarrow \psi$. Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. By hypothesis, $\mathcal{M}, w \models \varphi \rightarrow \psi$, that is $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$. Since, by hypothesis, $\mathcal{M}, w \models \varphi$ we must have $\mathcal{M}, w \models \psi$. Since $\mathcal{M}$ and w are arbitrary, $\models \psi$.

$(Gen_@)$ Suppose $\models \psi$. Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Thus $\mathcal{M}, v \models \psi$ where $V(i) = v$. Hence, $\mathcal{M}, w \models @_i\psi$. Therefore, $\models @_i\psi$.

$(Gen_\Box)$ Suppose $\models \psi$. Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Thus, $\mathcal{M}, v \models \psi$ for all v such that wRr. Hence, $\mathcal{M}, w \models \psi$. Therefore, $\models \Box\psi$.

(BG) Suppose $\models (@_i\Diamond j \rightarrow @_j\varphi)$. Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. By hypothesis, for $V(i) = v$ and $V(j) = z$, if $\mathcal{M}, v \models \Diamond j$ then $\mathcal{M}, z \models \varphi$. Since j is arbitrary, different from i and not occurring in φ, z can be any world accessible from v, i.e., vRz and, therefore, $\mathcal{M}, v \models \Box\varphi$.

$(Subs_\approx)$ Suppose that $\models \varphi \rightarrow (t \approx t')$. Let $\mathcal{M}$ be an algebraic hybrid structure and $w \in W$. Suppose that $\mathcal{M}, w \models \varphi$. Hence, by hypothesis, $\mathcal{M}, w \models t \approx t'$. Thus $A_w \models t \approx t'$. Thus,$A_w \models \bar{s}(t \approx t')$. Therefore, $\models \varphi \rightarrow \bar{s}(t \approx t')$

$(Subs)$ This is proved by induction on the structure of formulas. The base cases, for $\varphi = i$, $i \in$ NOM, or $\varphi = t_1 \approx t_2$ are immediate as nominals are replaced by nominals and variables by rigidified terms. Now consider $\varphi = \neg\psi$ and suppose, as an induction hypothesis, that for all structures $\mathcal{M}$ and $w \in W$, $\mathcal{M}, w \models \psi$ iff $\mathcal{M}, w \models \psi'$. Then, $\mathcal{M}, w \models \neg\psi$ iff $\mathcal{M}, w \not\models \psi$ which, by induction hypothesis, equivales to $\mathcal{M}, w \not\models \psi'$, and therefore $\mathcal{M}, w \models \neg\psi'$. The remaining cases are similar.

Theorem 1 (Soundness) *Every theorem of K_τ is valid. i.e., for any formula $\varphi \in$ Fm(τ),*

$$\Vdash \varphi \;\Rightarrow\; \models \varphi$$

Proof The proof follows by induction using the previous two lemmas.

As a consequence, by Lemma 3, we have

$$\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$$

Actually, $\Gamma \vdash \varphi$ iff there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Vdash \bigwedge \Gamma_0 \rightarrow \varphi$. By soundness, $\models \bigwedge \Gamma_0 \rightarrow \varphi$. Hence, $\bigwedge \Gamma_0 \models \varphi$; which implies $\Gamma \models \varphi$.

We shall now turn to prove completeness of the proposed axiomatisation. The following definitions and results are relevant to establish the envisaged theorem.

Definition 13 Let $\Gamma \subseteq \mathrm{Fm}(\tau)$.

- Γ is said to be K_τ-*inconsistent* if, $\Gamma \vdash \varphi$ for any $\varphi \in \mathrm{Fm}(\tau)$. Otherwise we say that Γ is K_τ-*consistent*.
- Γ is *maximal* K_τ-consistent if Γ is consistent and any set of formulas that properly extends Γ is inconsistent.
- Γ is *named* if it contains at least one nominal.
- Γ is $\Diamond$-*saturated* if for all $@_i \Diamond \varphi \in \Gamma$, there is a nominal j such that $@_i \Diamond j$ and $@_j \varphi$ belongs to Γ.

Lemma 9 *Let* $\Gamma \subseteq \mathrm{Fm}(\tau)$. *Then*

(1) Γ *is inconsistent iff there is a formula* φ *such that* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \neg\varphi$.
(2) $\varphi \in \Gamma$ *then* $\Gamma \vdash \varphi$.
(3) $\Gamma \cup \{\varphi\}$ *is inconsistent iff* $\Gamma \vdash \neg\varphi$.
(4) If Γ *is maximal consistent then,*

$$\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma.$$

Proof (1) Suppose that there is a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Vdash \bigwedge \Gamma_0 \to \varphi$ and $\Vdash \bigwedge \Gamma_0 \to \neg\varphi$. Let ψ be any formula. Then $\Vdash \bigwedge \Gamma_0 \to \psi$. Therefore $\Gamma \vdash \psi$. The converse is obvious.

(2) Since $\Vdash \varphi \to \varphi$ and $\varphi \in \Gamma$, by (*MP*) $\Gamma \vdash \varphi$.

(3) Suppose that $\Gamma \cup \{\varphi\}$ é inconsistent. Then $\Gamma \cup \{\varphi\} \vdash \neg\varphi$. Hence, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Vdash \bigwedge \Gamma_0 \to (\varphi \to \neg\varphi)$. Thus, $\Gamma \vdash \neg\varphi$. To see the converse, suppose that $\Gamma \cup \{\varphi\}$ is consistent. If $\Gamma \vdash \neg\varphi$ then we have $\Gamma \cup \{\varphi\} \vdash \neg\varphi$ and $\Gamma \cup \{\varphi\} \vdash \varphi$ which is an absurd.

(4) Suppose that $\varphi \notin \Gamma$. Hence $\Gamma \cup \{\varphi\}$ is inconsistent. Thus $\Gamma \vdash \neg\varphi$ which is absurd since Γ is consistent.

We now prove that every K_τ-consistent set of formulas can be extended to a named, $\Diamond$-saturated, maximal K_τ-consistent set.

Lemma 10 *Let* $(i_n)_{n\in\mathbb{N}}$ *a countable infinite set of new nominals,* $\overline{\tau}$ *be the new signature obtained by extending* Σ *and* NOM, *and* $K_{\overline{\tau}}$ *the correspondent equational hybrid logic (Note that, by substitution rule,* $K_{\overline{\tau}}$ *is a conservative extension of* K_τ*). Every* K_τ*-consistent set of formulas* Γ *can be extended to a named,* $\Diamond$*-saturated, maximal* $K_{\overline{\tau}}$*-consistent set.*

Proof Let Γ be a K_τ-consistent set of formulas and consider $(\varphi_n)_{n\in\mathbb{N}}$ an enumeration of all formulas in $\mathrm{Fm}(\tau)$. The set Γ^* is defined as $\bigcup_{n\in\mathbb{N}} \Gamma^n$, with

$$\Gamma^0 = \Gamma \cup \{i_0\};$$

$$\Gamma^{n+1} = \begin{cases} \Gamma^n, & \text{if } \Gamma^n \cup \{\varphi_n\} \text{ is inconsistent} \\ \Gamma^n \cup \{\varphi_n, @_i \Diamond i_m, @_{i_m}\psi\}, & \text{if } \varphi_n = @_i \Diamond \psi \text{ and } \Gamma^n \cup \{\varphi_n\} \text{ is consistent} \\ \Gamma^n \cup \{\varphi_n\}, & \text{otherwise} \end{cases}$$

where i_m is the first new nominal not occurring in Γ^n or in φ_n.

We first prove by induction that Γ^* is $K_{\overline{\tau}}$-consistent.

Suppose that Γ^0 is not consistent. Let $\varphi \in \mathrm{Fm}(\tau)$. Then $\Gamma \cup \{i_0\} \vdash \varphi$. Hence, by the deduction theorem, $\Gamma \vdash_{\overline{\tau}} i_0 \rightarrow \varphi$. Since i_0 does not occur in $\Gamma \cup \{\varphi\}$, by the rule $(Name')$, $\Gamma \vdash \varphi$.

Therefore, $\Gamma \vdash \varphi$ for any $\varphi \in \mathrm{Fm}(\tau)$, which is an absurd since Γ is consistent.

Suppose now that Γ^n is $K_{\overline{\tau}}$-consistent and consider φ_n of the form $@_i\Diamond\psi$ (the other cases are trivial). Suppose that $\Gamma^n + 1$ is not consistent. Let $\varphi \in \mathrm{Fm}(\tau)$. Then, $\Gamma^n \cup \{@_i\Diamond\psi, @_i\Diamond i_m, @_{i_m}\psi\} \vdash \varphi$. Hence, by (DDT), $\Gamma^n \cup \{@_i\Diamond\psi\} \vdash (@_i\Diamond i_m \wedge @_{i_m}\psi) \rightarrow \varphi$. By $(Paste_\Diamond)$, $\Gamma^n \cup \{@_i\Diamond\psi\} \vdash @_i\Diamond\psi \rightarrow \varphi$. Finally, by (DDT) again, $\Gamma^n \cup \{@_i\Diamond\psi\} \vdash \varphi$.

Since Γ^n is $K_{\overline{\tau}}$-consistent for $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} \Gamma^n$ is also $K_{\overline{\tau}}$-consistent.

We now prove that Γ^* is maximal. Conversely, suppose Γ^* is not maximal, that is, exists a formula $\varphi \notin \Gamma^*$ such that $\Gamma^* \cup \{\varphi\}$ is $K_{\overline{\tau}}$-consistent. Then $\varphi = \varphi_n$, for some $n \in \mathbb{N}$, and $\Gamma^n \cup \{\varphi_n\}$ is consistent. Consequently, $\varphi_n \in \Gamma^{n+1}$ which is an absurd since we assumed that $\varphi \notin \Gamma^*$.

In the sequel, given a K_τ-consistent set of formulas Γ, Γ^* will denote the maximal, named, $\Diamond$-saturated, and consistent extension of Γ, as defined in the proof of Lemma 10.

Lemma 11 *Let Γ be maximal consistent and named by k. Then for any formula φ,*

$$\varphi \in \Gamma \;\Leftrightarrow\; @_k\varphi \in \Gamma.$$

Proof Since $\Vdash k \rightarrow (\varphi \rightarrow @_k\varphi)$ $(Intro)$, we have that $\Gamma \vdash k \rightarrow (\varphi \rightarrow @_k\varphi)$, and $\Gamma \vdash \varphi \rightarrow @_k\varphi$, because $k \in \Gamma$. By hypothesis $\varphi \in \Gamma$, which entails $\Gamma \vdash @_k\varphi$, i.e., $@_k\varphi \in \Gamma$.

For the other direction, $\Vdash (k \wedge @_k\varphi) \rightarrow \varphi$ ($(Elim)$ in Proposition 2) entails $\Gamma \vdash (k \wedge @_k\varphi) \rightarrow \varphi$. Thus, since $k \in \Gamma$ and $@_k\varphi \in \Gamma$, we conclude, by applying (DDT) twice, that $\Gamma \vdash \varphi$.

Definition 14 Let Γ be a maximal, named, K_τ-consistent set of formulas. Binary relations $\sim_n$ and $\sim_r$, over NOM and $@T(\Sigma, X)$, respectively, are defined by

- $i \sim_n j \;\Leftrightarrow\; @_i j \in \Gamma,\; i, j \in \mathrm{NOM}$
- $t \sim_r t' \;\Leftrightarrow\; t \approx t' \in \Gamma,\; t, t' \in @T(\Sigma, X)$

Lemma 12 *The relations $\sim_n$ and $\sim_r$ are equivalence relations.*

Proof For $\sim_n$, reflexivity, symmetry and transitivity are direct consequence of rules $(Ref_@)$, $(Sym_@)$ and (Nom), respectively.

Consider now the case of $\sim_r$. Let $t_1, t_2, t_3 \in @T(\Sigma, X)$. Again reflexivity is immediate by rule $(Ref_@)$. For symmetry and transitivity, we reason

- *Symmetry*. Suppose $t_1 \sim_r t_2$. Then $t_1 \approx t_2 \in \Gamma$. Hence, for some nominal $k \in \Gamma$ (which exists because $\sim_r$ is defined based in a named set of formulas), by Lemma 11, $@_k(t_1 \approx t_2) \in \Gamma$. By (MP) and $(EQSym)$, $@_k(t_2 \approx t_i) \in \Gamma$ and finally, by $(Agree_\approx)$, $t_2 \approx t_1 \in \Gamma$. i.e., $t_2 \sim_r t_1$.

– *Transitivity*. Suppose $t_1 \sim_r t_2$ and $t_2 \sim_r t_3$, then $t_1 \approx t_2 \in \Gamma$ and $t_2 \approx t_3 \in \Gamma$. Hence, for some nominal $k \in \Gamma$, by Lemma 11, we have $@_k(t_1 \approx t_2) \in \Gamma$ and $@_k(t_2 \approx t_3) \in \Gamma$. By (MP) and $(EQTrans)$, $@_k(t_1 \approx t_3) \in \Gamma$ and we conclude, by $(Agree_\approx)$, that $t_1 \approx t_3 \in \Gamma$. i.e., $t_1 \sim_r t_3$.

Moreover,

Lemma 13 *Let* $f \in \Sigma_n$ *and* $t_k, t'_k \in @T(\Sigma, X)$, $k = 1, \ldots, n$ *and* $\Gamma \subseteq \text{Fm}(\tau)$ *maximal consistent named. Then, if* $t_k \sim_r t'_k$, $k = 1, \ldots, n$ *then for any nominal* $i \in \text{NOM}$

$$@_i f(t_1, \ldots, t_n) \sim_r @_i f(t'_1, \ldots, t'_n)$$

Proof Suppose that $t_k \sim t'_k$, for $k = 1, \ldots n$. Then $t_k \approx t'_k \in \Gamma$, for $k = 1, \ldots n$. By $(Agree_\approx)$, we have $@_i(t_k \approx t'_k) \in \Gamma$, for $k = 1, \ldots n$. Hence, by (MP) and $(EQFun)$ we have $@_i(f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n)) \in \Gamma$. Therefore, by $K_{@\approx}$, we have $@_i f(t_1, \ldots, t_n) \approx @_i f(t'_1, \ldots, t'_n) \in \Gamma$. i.e., $@_i f(t_1, \ldots, t_n) \sim_r @_i f(t'_1, \ldots, t'_n)$.

To prepare the grounds for the completeness proof, we define below the canonical structure. As usual, $\sim_n$ (respectively, $\sim_r$) equivalence classes are denoted by $|i|$ and $|@_i t|$ for each nominal i and each rigidified term $@_i t$, respectively.

Definition 15 Let Γ be a maximal named, $\Diamond$-saturated, K_τ-consistent set of formulas. Then, the *canonical structure* $\mathcal{M} = ((W^\Gamma, R^\Gamma, (A^\Gamma_{|i|})_{|i| \in W^\Gamma}), V^\Gamma)$ is defined as follows:

- $W^\Gamma = \{|i| : i \text{ is a nominal}\}$
- $|i| R^\Gamma |j|$ iff $@_i \Diamond j \in \Gamma$
- Each $A^\Gamma_{|i|}$ in $(A^\Gamma_{|i|})_{|i| \in W^\Gamma}$ is an algebra over the carrier $A^\Gamma = \{|t| : t \in @T(\Sigma, X)\}$
- $V^\Gamma(i) = |i|$, for each nominal i
- $f_{A^\Gamma_{|i|}}(|t_1|, \ldots, |t_n|) = |@_i f(t_1, \ldots, t_n)|$, for each $f \in \Sigma_n$ and $t_k, t'_k \in @T(\Sigma, X)$, $k = 1, \ldots, n$. In particular, for constants, $c_{A^\Gamma_{|i|}} = |@_i c|$.

Let us briefly check this definition, in particular that R^Γ is well defined. Suppose $i' \in |i|$, then $@_i i' \in \Gamma$ so, if $@_i \Diamond j \in \Gamma$, by (Nom), $@_{i'} \Diamond j \in \Gamma$. Now suppose $j' \in |j|$, then $@_j j' \in \Gamma$ so, if $@_i \Diamond j \in \Gamma$, by $(Bridge)$, $@_i \Diamond j'$.

The following results are relevant for proving Lemma 16 below, which plays a main role in the completeness proof.

Lemma 14 *If* Γ *is a maximal* K_τ*-consistent set of formulas, then the relation* $\sim_r$ *is a fully invariant congruence. i.e., for every substitution* $s : X \longrightarrow \text{Term}(\tau)$, $t, t' \in @T(\Sigma, X)$

$$t \sim_r t' \;\Rightarrow\; \bar{s}(t) \sim_r \bar{s}(t')$$

where $\bar{s} : \text{Term}(\tau) \longrightarrow \text{Term}(\tau)$ *is the canonical extension of* s *to terms.*

Proof

$$\begin{aligned}
&t \sim_r t' \\
&\quad \Leftrightarrow t \approx t' \in \Gamma \\
&\quad \Leftrightarrow \Gamma \vdash t \approx t' \\
&\quad \Leftrightarrow \text{there is a finite } \Gamma_0 \subseteq \Gamma s.t. \Vdash \bigwedge \Gamma_0 \to t \approx t' \\
&\quad \Rightarrow \text{there is a finite } \Gamma_0 \subseteq \Gamma s.t. \Vdash \bigwedge \Gamma_0 \to \bar{s}(t \approx t'), \text{ by } (Subs_{\approx}) \\
&\quad \Leftrightarrow \Gamma \vdash \bar{s}(t \approx t') \\
&\quad \Leftrightarrow \bar{s}(t) \sim_r \bar{s}(t')
\end{aligned}$$

Recall that, for each assignment $g : X \to A^{\Gamma^*}_{|i|}$, there is a substitution $s : X \to @T(\Sigma, X)$ such that $g(x) = |s(x)|$. Its extension to terms is considered in the following lemma.

Lemma 15 *For any $t \in \mathrm{Term}(\tau)$,*

$$[t]^{\mathcal{M}^{\Gamma^*},|i|,g} = |\bar{s}(@_i t)|$$

Proof The proof proceeds by induction on the structure of terms. Relation $\sim_r$ will be abbreviated to $\sim$ to simplify notation.

- $t = x$, *for a variable* x

$$[x]^{\mathcal{M}^{\Gamma^*},|i|,g} = g(x) = |\bar{s}(x)| = |@_i \bar{s}(x)| = |\bar{s}(@_i x)|$$

  using axiom $(Loc_{\approx})$.
- $t = c$, *for a constant* c

$$[c]^{\mathcal{M}^{\Gamma^*},|i|,g} = |@_i c| = |\bar{s}(@_i c)|$$

- $t = f(t_1, \ldots, t_n)$, *for a function symbol* f

$$\begin{aligned}
&[f(t_1, \ldots, t_n)]^{\mathcal{M}^{\Gamma^*},|i|,g} \\
&\quad = f_{A^{\mathcal{M}}_{|i|}}([t_1]^{\mathcal{M}^{\Gamma^*},|i|,g}, \ldots, [t_n]^{\mathcal{M}^{\Gamma^*},|i|,g}) \\
&\quad = f_{A^{\mathcal{M}}_{|i|}}(|\bar{s}(@_i t_1)|_{\sim}, \ldots, |\bar{s}(@_i t_n)|) \qquad \text{(by induction)} \\
&\quad = |@_i f(\bar{s}(@_i t_1), \ldots, \bar{s}(@_i t_n))| \\
&\quad = |\bar{s}(@_i f(@_i t_1, \ldots, @_i t_n))| \\
&\quad = |\bar{s}(@_i f(t_1, \ldots, t_n))|
\end{aligned}$$

  The last step follows from $\vdash @_i f(t_1, \ldots, t_n) \approx @_i f(@_i t_1, \ldots @_i t_n)$, which is derived as follows:

$$\dfrac{\dfrac{\dfrac{\overline{@_i x \approx x}\ (Loc_{\approx})}{@_i @_i t_k \approx @_i t_k,\ k=1,\dots,n}\ (Subs)}{@_i(@_i t \approx t_k),\ k=1,\dots,n}\ (K_{@\approx})}{@_i f(t_1,\dots,t_n) \approx\ @_i f(@_i t_1,\dots,@_i t_n)}\ (EQFun)$$

Then, by definition of $\sim$, $@_i f(t_1,\dots,t_n) \sim_r @_i f(@_i t_1,\dots @_i t_n)$. Thus, $\bar{s}(@_i f(t_1,\dots,t_n)) \sim_r \bar{s}(@_i f(@_i t_1,\dots @_i t_n))$.

– $t = @_j t_0$, *for a nominal* j

$$\begin{aligned}
&[@_j t_0]^{\mathcal{M}^{\Gamma^*},|i|,g}\\
&\quad = [t_0]^{\mathcal{M}^{\Gamma^*},|j|,g}\\
&\quad = |\bar{s}(@_j t_0)| \qquad \text{(by induction)}\\
&\quad = |\bar{s}(@_i @_j t_0)|
\end{aligned}$$

The last step comes from $\vdash @_i @_j t_0 \approx @_j t_0$, which, by definition of $\sim$, entails $@_i @_j t_0 \sim @_j t_0$.

Lemma 16 (Truth Lemma) *Let Γ be a K_τ-consistent set of formulas. Then, for every nominal i and every formula φ,*

$$\mathcal{M}^{\Gamma^*}, |i| \models \varphi \quad \Leftrightarrow \quad @_i\varphi \in \Gamma^*$$

Proof The proof proceeds by induction on the complexity of φ.

Let $\varphi = j$. We have that

$$\mathcal{M}^{\Gamma^*}, |i| \models j \textit{ iff } |i| = |j| \textit{ iff } @_i j \in \Gamma^*.$$

Let $\varphi = t_1 \approx t_2$. We know that, $\mathcal{M}^{\Gamma^*}, |i| \models \varphi$ iff for any assignment $g : X \to A^{\Gamma^*}_{|i|}$ $[t_1]^{\mathcal{M}^{\Gamma^*},|i|,g} = [t_2]^{\mathcal{M}^{\Gamma^*},|i|,g}$. This implies that for any assignment $g : X \to A^{\Gamma^*}_{|i|}$ $|\bar{s}(@_i t_1)| = |\bar{s}(@_i t_2)|$, with $s : X \to @T(\Sigma, X)$ is such that $g(x) = |s(x)|$. In particular, by taking $g(x) = |x|$ $(s(x) = x)$ we have $|@_i t_1| = |@_i t_2|$. i.e., $@_i t_1 \approx @_i t_2 \in \Gamma^*$. And finally, by $(K_{@\approx})$ and (MP), we have $@_i(t_1 \approx t_2) \in \Gamma^*$. Conversely, suppose that $@_i(t_1 \approx t_2) \in \Gamma^*$. Hence, $@_i t_1 \approx @_i t_2 \in \Gamma^*$. Equivalently, $@_i t_1 \sim_r @_i t_2$. Let $g : X \to A^{\Gamma^*}_{|i|}$ and $s : X \to @T(\Sigma, X)$ the substitution such that $g(x) = |s(x)|$. By Lemma 14, $\bar{s}(@_i t_1) \sim_r \bar{s}(@_i t_2)$. That is, $|\bar{s}(@_i t_1)| = |\bar{s}(@_i t_2)|$. Finally by Lemma 15, $[t_1]^{\mathcal{M}^{\Gamma^*},|i|,g} = [t_2]^{\mathcal{M}^{\Gamma^*},|i|,g}$.

Let $\varphi = \Diamond\psi$. Assume $\mathcal{M}^{\Gamma^*}, |i| \models \Diamond\psi$. Then there is a nominal j such that $|i|R|j|$ and $\mathcal{M}^{\Gamma^*}, |j| \models \psi$. Since $|i|R|j|$, $@_i j \in \Gamma^*$, and also we have $@_i\psi \in \Gamma^*$ (by induction hypothesis), then by $(Bridge)$, $@_i\Diamond\psi \in \Gamma^*$.

Conversely, assume that $@_i\Diamond\psi \in \Gamma^*$. Then, by maximality of Γ^*, $@_i\psi \in \Gamma^*$ and $@_i\Diamond j \in \Gamma^*$, for some nominal j. Then $|i|R^{\Gamma^*}|j|$ and since, by induction hypothesis, $\mathcal{M}^{\Gamma^*}, |j| \models \psi$, $\mathcal{M}^{\Gamma^*}, |i| \models \Diamond\psi$.

Let $\varphi = \neg\psi$. Assume $\mathcal{M}^{\Gamma^*}, |i| \models \neg\psi$. Then $\mathcal{M}^{\Gamma^*}, |i| \not\models \psi$ and, by induction hypothesis, $@_i\psi \notin \Gamma^*$. Since Γ^* is maximal and consistent, $\neg @_i\psi \in \Gamma^*$ and, by $(Selfdual_@)$, $@_i\neg\psi \in \Gamma^*$.

Conversely, suppose $@_i\neg\psi \in \Gamma^*$. Then, by $(Selfdual_@)$, $\neg @_i\psi \in \Gamma^*$. Since Γ^* is consistent, $@_i\psi \notin \Gamma^*$ and, by induction hypothesis, $\mathcal{M}, |i| \not\models \psi$ and consequently $\mathcal{M}, |i| \models \neg\psi$.

Let $\varphi = @_j\psi$. Assume $\mathcal{M}^{\Gamma^*}, |i| \models @_j\psi$, then $\mathcal{M}^{\Gamma^*}, |j| \models \psi$ and, by induction hypothesis, $@_j\psi \in \Gamma^*$. By $(Agree)$, $@_i@_j\psi \in \Gamma^*$.

Conversely, suppose $@_i@_j\psi \in \Gamma^*$. Then, by $(Agree)$, $@_j\psi \in \Gamma^*$. By hypothesis, $\mathcal{M}, |j| \models \psi$ and consequently $\mathcal{M}, |i| \models @_j\psi$.

Finally, let $\varphi = \psi_1 \wedge \psi_2$. Assume $\mathcal{M}^{\Gamma^*}, |i| \models \psi_1 \wedge \psi_2$, then $\mathcal{M}^{\Gamma^*}, |i| \models \psi_1$ and $\mathcal{M}^{\Gamma^*}, |i| \models \psi_2$. By induction hypothesis, $@_i\psi_1 \in \Gamma^*$ and $@_i\psi_2 \in \Gamma^*$ then, by $(Conj)$ $@_i(\psi_1 \wedge \psi_2) \in \Gamma^*$.

Conversely, suppose $@_i(\psi_1 \wedge \psi_2) \in \Gamma^*$. Then, by $(Conj)$, $@_i\psi_1 \in \Gamma^*$ and $@_i\psi_2 \in \Gamma^*$. By hypothesis, $\mathcal{M}, |i| \models \psi_1$ $\mathcal{M}, |i| \models \psi_2$ and then $\mathcal{M}, |i| \models \psi_1 \wedge \psi_2$.

Lemma 17 *Let Γ be a consistent set of formulas. Then, there is a nominal k such that for every $\varphi \in \Gamma^*$,*

$$\mathcal{M}^{\Gamma^*}, |k| \models \varphi$$

Proof Let Γ be a consistent set of formulas and $\varphi \in \Gamma^*$. Since Γ^* is named, there is a nominal k in Γ^*. Then, by Lemma 11, $@_k\varphi \in \Gamma^*$. Therefore, by Lemma 16, $\mathcal{M}^{\Gamma^*}, |k| \models \varphi$.

Theorem 2 (Completeness) *Given a hybrid equational similarity type τ, let φ be a formula and Γ a set of formulas. Then*

$$\Gamma \models \varphi \;\Leftrightarrow\; \Gamma \vdash \varphi$$

Proof We have already show the implication $\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \models \varphi$.

If Γ is inconsistent, the converse is immediate. Assume Γ is consistent and suppose that φ is not derivable from Γ. Then, by Lemma 9 (3), $\Gamma \cup \{\neg\varphi\}$ is consistent. Consider Δ a named and $\Diamond$-saturated, maximal consistent extension of $\Gamma \cup \{\neg\varphi\}$. By Lemma 17, there is a nominal k such that $\mathcal{M}^{\Delta}, |k| \models \Delta$ and $\mathcal{M}^{\Delta}, |k| \models \neg\varphi$. Hence, $\mathcal{M}^{\Delta}, |k| \not\models \varphi$. Therefore, $\Gamma \not\models \varphi$

5 Conclusions and Future Work

This paper introduced an axiomatisation for hybrid equational logic and established its soundness and completeness. The proposed approach can be regarded as a fragment of the first order hybrid logic discussed in Braüner (2011). The focus on the *equational* case was already explained: in the specification method for reconfigurable systems proposed in Madeira et al. (2011) (see also Manzano et al. (2012)), equational logic is found most appropriate to specify each local configuration. On its turn, the system's reconfigurations are expressed by a modal language over a Kripke frame whose states are exactly the local equational specifications. The hybrid component relates both levels, namely by 'indexing' local properties to specific states.

Reference Madeira et al. (2011) provides a detailed account of this method. For the moment, however, a small, toy example may help to illustrate the kind of systems we are concerned with. Consider a calculator with two possible configurations: in one of them an operation $\star$ stands for addition of natural numbers, whereas in the other it corresponds to multiplication. A special button *shift* leads from one configuration to the other.

This calculator may be regarded as a transition system that alternates between *sum* and *multiplication* modes through an event (modality) *shift*. Each of its states is associated to a Σ-algebra, where Σ has the following operation symbols $c :\rightarrow nat$, $s : nat \rightarrow nat$, $p : nat \rightarrow nat$ and $\star : nat \times nat \rightarrow nat$. Global properties are expressed equationally; for example

$$p(s(n)) \approx n$$

to characterise p as the predecessor function, or

$$\star(n, k) \approx \star(k, n) \quad \text{or} \quad \star(n, \star(k, l)) \approx \star(\star(n, k), l)$$

to express $\star$ commutativity and associativity, respectively.

On the other hand, the specification of local properties, i.e., properties that hold in particular modes, entails the need for the introduction of a nominal, say NOM $= \{ref\}$, to identify, for instance, the mode where $\star$ plays the role of a sum. Hence, we are able to state, for example

$$@_{ref} \star (n, c) \approx n \quad \text{and} \quad @_{ref} \star (n, s(c)) \approx s(n)$$

or

$$@_{ref}[shift] \star (n, c) \approx c \quad \text{and} \quad @_{ref}[shift] \star (n, s(c)) \approx n$$

Finally, alternation between the two operating modes is captured by modal properties; for example,

$$\neg @_{ref} \langle shift \rangle ref \quad \text{and} \quad @_{ref}[shift][shift]ref$$

Note that, since in this specification method local properties are functional, predicate symbols were not considered in the language discussed here. They can, however, be added, in a standard way, resorting to equality tests.

Also interesting is to note how propositional modal logic can be translated to hybrid equational logic. Sketching the construction, let Σ be an algebraic signature consisting of distinct constants $\{c\} \cup \{d_p : p \in \text{PROP}\}$, PROP denoting the set of propositional variables. A translation $\bar{\alpha} : \text{Form}(\tau) \rightarrow \text{Fm}(\tau)$ is given as the natural extension (i.e., compatible with boolean operations) of $\alpha : \text{PROP} \cup \text{NOM} \rightarrow \text{Eq}_{\Sigma}(X) \cup \text{NOM}$ defined by

- for each propositional symbol p, $\alpha(p) = c \approx d_p$;
- and $\alpha(i) = i,\ i \in \text{NOM}$.

Let $\mathcal{M} = ((W, R, (A_w)_{w \in W}), V)$ be an algebraic hybrid structure and define the propositional hybrid model $\bar{\mathcal{M}} = (W, R, \bar{V})$ where $\bar{V}(i) = V(i), i \in \text{NOM}$ and $\bar{V}(p) = \{w \in W : A_w \models \alpha(p)\}$. Then, for any hybrid propositional formula φ

$$\mathcal{M}, w \models \varphi \Leftrightarrow \bar{\mathcal{M}}, w \models \alpha(\varphi)$$

Reciprocally, let $\mathcal{H} = (W, R, V)$ be a propositional hybrid model. Let A be a set with at least two elements, say a, b. Define the algebraic hybrid structure $\bar{\mathcal{H}} = ((W, R, (A_w)_{w \in W}), \bar{V})$, where $\bar{V}(i) = V(i), i \in \mathrm{NOM}$, $(A_w)_c = a$ for any $w \in W$ and

$$(A_w)_{d_p} = \begin{cases} a, & \text{if } w \in V(p) \\ b, & \text{otherwise.} \end{cases}$$

Then,

$$\mathcal{H}, w \models \varphi \;\Leftrightarrow\; \bar{\mathcal{H}}, w \models \alpha(\varphi).$$

There are other approaches that add algebraic features to hybrid logic. We would like to mention Tzanis (2005) which adds algebraic structure to nominals and develops all the hybrid machinery in this case, providing, in particular a notion of bisimulation. Reference Goranko and Vakarelov (1998) discusses another kind of algebraic generalization of modal logic. As a further remark, it would be interesting to work on a proof of completeness in Henkin's style to the extension of hybrid logic in which the set of nominals is endowed with an algebraic structure, as introduced in Tzanis (2005). Although these algebraic features of Tzanis' logic are placed at different level, it is worth to study a combination of both algebraic aspects.

The approach proposed in this paper to combine hybrid and equational logic does not follow the standard hybrid extensions of the orthodox quantified modal logic (as in, for example, Fitting and Mendelsohn (1998) and Garson (1984)). Such extensions lead to quite complex logics, often of difficult application. Moreover almost all of them do not allow functional symbols (Braüner 2011; Blackburn and Cate 2006).

In our perspective, simpler logics are worth to explore in applications since they pave the way to developing efficient (semi)automatic provers. Combining well-known proof procedures for equational logic (e.g., rewriting) with provers for hybrid logic is a topic we intend to explore in the future. This is also related to other relevant research issues not addressed here, in particular decidability and computational complexity of the satisfiability problem (Areces et al. 2001).

But the main challenge driving our current work is methodological: how does the approach proposed in this paper scales? Or, to put it more rigorously, can an axiomatisation for an 'hybridised' logic be obtained through a systematic extension of an axiomatisation for basic hybrid logic with elements of the calculus of the logic to 'hybridise'?. The equational case seems encouraging. A proper, more general answer, however, needs to be sought at a more general level. The institutional framework, in which our research programme on 'hybridisation' Martins et al. (2011) is being conducted, provides the arena for the forthcoming steps.

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References

Areces, C., Blackburn, P., & Marx, M. (2001). Hybrid logics: Characterization, interpolation and complexity. *Journal of Symbolic Logic*, *66*(3), 977–1010.

Blackburn, P. (2000). Representation, reasoning, and relational structures: A hybrid logic manifesto. *Logic Journal of IGPL*, *8*(3), 339–365.

Blackburn, P., & ten Cate, B. D. (2006). Pure extensions, proof rules, and hybrid axiomatics. *Studia Logica*, *84*(2), 277–322.

Blackburn, P., de Rijke, M., & Venema, Y. (2001). *Modal logic*. Cambridge: Cambridge University Press.

Braüner, T. (2011). *Hybrid logic and its proof theory*. Berlin: Springer.

Braüner, T. (2005). Natural deduction for first-order hybrid logic. *Journal of Logic, Language and Information*, *14*, 173.

Braüner, T., & de Paiva, V. (2006). Intuitionistic hybrid logic. *Journal of Applied Logic*, *4*(3), 231–255.

Fitting, M., & Mendelsohn, R. L. (1998). *First-order modal logic*. Dordrecht: Kluwer.

Garson, J. W. (1984). Quantification in modal logic. In D. M. Gabbay & F. Guenthner (Eds.) *Handbook of philosophical logic: Volume III: Extensions of classical logic* (pp. 249–307). Dordrecht: Reidel.

Goranko, V., & Vakarelov, D. (1998). Modal logic and universal algebra i: Modal axiomatizations of structures. *Advances in Modal Logic*, 247–274.

Grätzer, G. (1979). *Universal algebra* (2nd ed.). Berlin: Springer.

Indrzejczak, A. (2007). Modal hybrid logic. *Logic and Logical Philosophy*, *16*, 147–257.

Madeira, A., Faria, J. M., Martins, M. A., & Barbosa, L. S. (2011). Hybrid specification of reactive systems: An institutional approach. In *9th international conference on software engineering and formal methods (SEFM'11)*. Springer Lecture Notes Computer Science 7041, pp. 250–259.

Manzano, M., Martins, M.A., & Huertas, A. (2012). Equational hybrid type theory. In T. Bolander, T. Braüner, S. Ghilardi & L. Moss (Eds.) *In proceedings of 9th advances in modal logic (AiML'12) short presentations* (pp. 37–41). Copenhagen.

Martins, M. A., Madeira, A., & Barbosa, L. S. (2012). Towards a semantics for infinitary equational hybrid logic. In T. Bolander, T. Braüner, S., Ghilardi & L. Moss (Eds.) *In proceedings of 9th advances in modal logic (AiML'12) short presentations* (pp. 42–46). Copenhagen.

Martins, M. A., Madeira, A., Diaconescu, R., & Barbosa, L. S. (2011). Hybridization of institutions. In *Fourth international conference on algebra and Coalgebra in computer science (CALCO'11)*, pp. 283–297. Springer Lecture Notes in Computer Science (6859).

Tzanis, E. (2005). Algebraizing hybrid logic. Master's thesis, Faculty Faculty of Science Institute/dept. FNWI/FGw: Institute for Logic, Language and Computation (ILLC).