arXiv:1507.03219v1 [cs.LO] 12 Jul 2015

Continuity as a computational effect

Renato Neves[a], Luís S. Barbosa[a], Dirk Hofmann[b], Manuel A. Martins[b]

[a] *INESC TEC (HASLab) & Universidade do Minho, Portugal*
rjneves@inescporto.pt, lsb@di.uminho.pt
[b] *CIDMA - Dep. of Mathematics, Universidade de Aveiro, Portugal*
{dirk,martins}@ua.pt

Abstract

The original purpose of component–based development was to provide techniques to master complex software, through composition, reuse and parametrisation. However, such systems are rapidly moving towards a level in which software becomes prevalently intertwined with (continuous) physical processes. A possible way to accommodate the latter in component calculi relies on a suitable encoding of continuous behaviour as (yet another) computational effect.

This paper introduces such an encoding through a monad which, in the compositional development of hybrid systems, may play a role similar to the one played by the 1+, powerset, and distribution monads in the characterisation of partial, non deterministic and probabilistic components, respectively. This monad and its Kleisli category provide a setting in which the effects of continuity over (different forms of) composition can be suitably studied.

Keywords: Monads, components, hybrid systems, control theory

---

1. Introduction

Motivation and objectives. Component-based software development is often explained through a visual metaphor: a palette of computational units, and a blank canvas in which they are dropped and interconnected by drawing wires abstracting different composition and synchronisation mechanisms. More and more, however, components are not limited to traditional information processing units, but encapsulate some form of interaction with physical processes. The resulting systems, referred to as *hybrid* [1], exhibit a complex dynamics in which *loci* of computation, coordination, and control of physical processes interact, become mutually constrained, and cooperate to achieve specific goals.

One way of looking at components, proposed in [2], emphasises an observational semantics, through a signature of observers and methods, making them amenable to a coalgebraic characterisation as (generalisations of) abstract Mealy machines. The resulting calculus is parametric on whatever behavioural model underlies a component specification. This captures, for example, partial, non deterministic or probabilistic evolution of a component's dynamics by encoding such behavioural effects as strong monads [3] — a pervasive mathematical structure with surprising applications in different areas of Computer Science (see *e.g.* [4, 5, 6, 7, 8]).

Indeed, each monad captures a specific behaviour, which becomes reflected in the corresponding component calculus. For example, the *maybe* monad ($1+$) introduces *partial* components; the *powerset* ($\mathcal{P}$) monad *non-deterministic* ones; and the *distribution* monad ($\mathcal{D}$) brings (discrete) *probabilistic* evolution into the scene. Can *continuous behaviour*, prevalent in hybrid systems and control theory, be encoded in a similar way, as (yet another) computational effect? Such is the question addressed here.

The use of monads to structure the denotational semantics of programming languages was proposed in the 80's, by E. Moggi [9, 4]. Later the concept was introduced in programming practice by P. Wadler [5], entailing a rigorous style of combining purely functional programs that mimic impure (side-)effects. The key idea is that monads permit to encode in abstract terms several kinds of computational effects, such as exceptions, state updating, nondeterminism or continuations. Such effects are represented by a type constructor (*i.e.*, an endofunctor in a suitable category) $\mathcal{M}$ so that computations producing values of type O are regarded as terms of type $\mathcal{M}O$. In this way, *values* and *computations* are explicitly distinguished and programs can be thought of as arrows $I \to \mathcal{M}O$ representing the computation of values of type O from values of type I, while producing some effect described by $\mathcal{M}$. Or, putting it in a different way, output values arise encapsulated (or embedded) in the effect specified by $\mathcal{M}$. As a generalised monoid, a monad comes equipped with an identity and an associative multiplication which, from a computational point of view, builds a (trivial) computation from a value, and captures the flattening of nested effects, respectively. Furthermore, if $\mathcal{M}$ is *strong* [5] additional machinery is available to distribute the computations effect over context.

The paper introduces a strong monad $\mathcal{H}$ that subsumes continuous behaviour and briefly explores its Kleisli category as the mathematical space in which the underlying behaviour can be isolated and its effect over different forms of composition studied. Again this parallels the role that the categories of partial functions, relations and stochastic matrices have as reasoning universes for component composition under the behavioural model provided, respectively, by monads $1+$, $\mathcal{P}$ and $\mathcal{D}$ [10, 11]. Similarly, this work may pave the way to the development of a calculus of *hybrid components*.

Document structure. After a brief detour on preliminaries and notation in Section 2, the continuous-evolution monad $\mathcal{H}$ is introduced in Section 3. Section 4 explores the corresponding Kleisli category $Kl\,\mathcal{H}$, characterising composition and some (co)limits. Finally, related work and possible future research directions are discussed in Section 5. Most of the proofs adopt a pointfree style in the spirit of the Bird-Meertens formalism [12].

$$\frac{f : X \to Y, g : Y \to Z}{g \cdot f : X \to Z}\ (\cdot) \qquad \frac{f : X \times Y \to Z}{\lambda f : X \to Z^Y}\ (\lambda)$$

$$\frac{f : X \to Y_1, g : X \to Y_2}{\langle f, g\rangle : X \to Y_1 \times Y_2}\ (\times) \qquad \frac{f : X_1 \to Y, g : X_2 \to Y}{[f, g] : X_1 + X_2 \to Y}\ (+)$$

$$\frac{f : X \to Y, A \subseteq X}{f_A : A \to Y}\ (\downarrow_l) \qquad \frac{f : X \to Y, \text{img } f \subseteq B}{f^B : X \to B}\ (\downarrow_r)$$

$$\text{with } f_A = f \cdot \iota \text{ (for } \iota : A \hookrightarrow X) \qquad \text{with } \iota \cdot f^B = f \text{ (for } \iota : B \hookrightarrow Y)$$

Figure 1: Continuity rules in $\mathbb{T}op$.

2. Preliminaries

As usual, we qualify as *continuous* a system whose output, for any given input, is a (continuous) evolution over time; *i.e.*, an arrow typed as

$$I \to \bigcup_{d \in [0,\infty]} O^{[0,d]}$$

where I, O are input and output universes, respectively, $O^{[0,d]}$ the space of continuous functions from $[0, d]$ to O (the *evolutions*), and $[0, d]$ a specific duration. This suggests the category $\mathbb{T}op$ of topological spaces and continuous functions as a suitable working environment for developing the envisaged results.

In the sequel, if the context is clear, a topological space will be denoted by its underlying set. Also, assume that spaces $X \times Y$, $X + Y$ correspond to the canonical product and coproduct of X, Y, respectively, and that for any $X \subseteq Y$, X has the subspace topology induced by Y. Finally, whenever Y is core-compact, space X^Y denotes the exponential topology [13]. $\mathbb{T}op$ is complete and cocomplete, so we will often resort to isomorphisms $\alpha_l : (X \times Y) \times Z \cong X \times (Y \times Z)$, and $sw : X \times Y \cong Y \times X$. Finally, $\mathbb{T}op$ provides a set of useful rules for showing continuity; Figure 1 sums up the ones used in the paper. Note that in rule (λ), Y must be core-compact [13] so that the evaluation function $ev : X^Y \times Y \to X$ is defined.

Notation. Universal arrows $X \to 1$ to the final object in $\mathbb{T}op$ will be denoted by !, and a function constantly yielding a value x by $\underline{x}$. Given two functions $f, g : X \to Y$, and a predicate p, we introduce a conditional expression $f \lhd p \rhd g : X \to Y$, defined by

$$(f \lhd p \rhd g)\, x = \begin{cases} f\, x & \text{if } p\, x \\ g\, x & \text{otherwise} \end{cases}$$

The continuous functions *minimum* $\curlywedge : \mathbb{R} \times (\mathbb{R} + 1) \to \mathbb{R}$ and *truncated subtraction* $\ominus : \mathbb{R} \times (\mathbb{R} + 1) \to \mathbb{R}$ play a key role in the sequel. They are defined as

$$
\begin{aligned}
r \curlywedge (i_1\, s) &= (\pi_1 \lhd (\leq) \rhd \pi_2)\,(r, s) & \qquad r \ominus (i_1\, s) &= ((-) \lhd (>) \rhd \underline{0})\,(r, s) \\
r \curlywedge (i_2\, \star) &= r & \qquad r \ominus (i_2\, \star) &= 0
\end{aligned}
$$

where $\leq, >$ are the usual ordering relations over the reals, and 1 introduces infinity. Finally, note also the continuous function $ev_r : X^Y \times (Y+1) \to (Y+1)$ defined as $ev_r(f, i_1\, y) = i_1 \cdot f\, y$ and $ev_r(f, i_2\, \star) = i_2\, \star$

3. $\mathcal{H}$: A monad for continuous evolution

As mentioned above, continuous systems can be regarded as arrows of type

$$I \to \bigcup_{d \in [0,\infty]} O^{[0,d]}$$

In order to define them in $\mathbb{T}op$ it is necessary to equip the target object with a suitable topology. However, since there is not a canonical topology able to relate functions with different domains, we consider that all evolutions have domain $\mathbb{R}_0$. This is possible when one notices that $[0, d]$ is a retract of $\mathbb{R}_0$ through the minimum function (the retraction)

$$\curlywedge_d : \mathbb{R}_0 \to [0, d]$$

Diagrammatically,

$$
\begin{tikzcd}
{[0,d]} \arrow[rr, dotted, "f^{\urcorner}"] & & X \\
& \mathbb{R}_0 \arrow[ul, "\curlywedge_d"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

Thus, each evolution $f : [0, d] \to X$ becomes $f \cdot \curlywedge_d : \mathbb{R}_0 \to X$. It is also important to take duration of evolutions into account. So the definition of continuous system becomes

$$I \to O^{\mathbb{R}_0} \times D$$

where $D = \mathbb{R}_0 + 1$, and for each evolution $f \in O^{\mathbb{R}_0}$, $f = f \cdot \curlywedge_d$. Note that, due to the topological coproduct in D, 'similar' systems always execute for a similar period of time, which renders impossible one to execute forever and not the other.

Definition 1. *$\mathcal{H} : \mathbb{T}op \to \mathbb{T}op$ is a mapping such that, for any objects $X, Y \in |\mathbb{T}op|$ and any continuous function $g : X \to Y$,*

$$
\begin{aligned}
&\mathcal{H}X = \{\ (f, d) \in X^{\mathbb{R}_0} \times D \mid f \cdot \curlywedge_d = f\ \} \\
&\mathcal{H}g = g \cdot \times id
\end{aligned}
$$

To see that the image of $\mathcal{H}g$ is contained in $\mathcal{H}Y$, note that $f = f \cdot \lambda_d$ entails $g \cdot f = g \cdot f \cdot \lambda_d$. Morever, since $(g\ .) : X^{\mathbb{R}_0} \to Y^{\mathbb{R}_0}$ is continuous, $\mathcal{H}g$ also is.

Theorem 1. *$\mathcal{H}$ is a functor.*

Proof. Functoriality follows from the equation

$$\mathcal{H}\, g = \big((\text{ - } \times D) \cdot (\text{ - })^{\mathbb{R}_0}\, g \big) \cdot \iota$$

for any continuous function $g : X \to Y$. $\square$

The crucial step now is to equip $\mathcal{H}$ with a monad structure

$$\eta : Id \to \mathcal{H}, \mu : \mathcal{H}^2 \to \mathcal{H}$$

First,

Definition 2. *Consider continuous functions $\pi_1 : X \times \mathbb{R}_0 \to X$, $i_1 \cdot \underline{0} : X \to D$. Then, define $\eta_X = \langle \lambda \pi_1, i_1 \cdot \underline{0} \rangle$*

Lemma 1. *η is a natural transformation,* i.e. *η_X is continuous for any space $X \in \mathbb{T}op$, and the diagram below commutes*

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
\mathcal{H}X \arrow[r, "\mathcal{H}h"'] & \mathcal{H}Y
\end{tikzcd}$$

for any continuous function $h : X \to Y$.

Proof. Function η_X is continuous because

$$\dfrac{\dfrac{\dfrac{\pi_1 : X \times \mathbb{R}_0 \to X}{\lambda \pi_1 : X \to X^{\mathbb{R}_0}}\ (\ \lambda\ )}{\langle \lambda \pi_1, i_1 \cdot \underline{0}\ \rangle : X \to X^{\mathbb{R}_0} \times D}\ (\ \times\ )}{\langle \lambda \pi_1, i_1 \cdot \underline{0}\ \rangle : X \to \mathcal{H}X}\ (\ \downarrow_r\ )$$

Then, diagram

$$\begin{tikzcd}
x \arrow[r, mapsto, "h"] \arrow[d, mapsto, "\eta_X"'] & h\, x \arrow[d, mapsto, "\eta_Y"] \\
(\underline{x}, i_1\, 0) \arrow[r, mapsto, dotted, "h\, \cdot \times id"'] & (\underline{h\, x}, i_1\, 0)
\end{tikzcd}$$

commutes because $h \cdot \underline{x} = \underline{h\, x}$. $\square$

The definition of μ is more demanding.

Definition 3. *Define the continuous function* $g : \mathcal{H}^2 X \times \mathbb{R}_0 \to X^{\mathbb{R}_0}$ *such that*

$$g = \pi_1 \cdot ev \cdot (id \times (\Yup \cdot sw)) \cdot \alpha_r$$

$$\big((\mathcal{H}X)^{\mathbb{R}_0} \times D\big) \times \mathbb{R}_0 \xrightarrow{\alpha_r} (\mathcal{H}X)^{\mathbb{R}_0} \times (D \times \mathbb{R}_0) \xrightarrow{id \times sw}$$

$$(\mathcal{H}X)^{\mathbb{R}_0} \times (\mathbb{R}_0 \times D) \xrightarrow{id \times \Yup} (\mathcal{H}X)^{\mathbb{R}_0} \times \mathbb{R}_0 \xrightarrow{ev}$$

$$\mathcal{H}X \xrightarrow{\pi_1} X^{\mathbb{R}_0}$$

and function $h : \mathcal{H}^2 X \times \mathbb{R}_0 \to \mathbb{R}_0$

$$h = \ominus \cdot sw \cdot (\pi_2 \times id)$$

$$\mathcal{H}^2 X \times \mathbb{R}_0 \xrightarrow{\pi_2 \times id} D \times \mathbb{R}_0 \xrightarrow{sw} \mathbb{R}_0 \times D \xrightarrow{\ominus} \mathbb{R}_0$$

Finally, define $fl_1 = \lambda(ev \cdot \langle g, h \rangle)$. *In pointwise notation,* fl_1 *is defined as* $fl_1(f, d) = ev \cdot \langle \pi_1 \cdot f \cdot \Yup_d, \ominus_d \rangle$, *which, since* $f = f \cdot \Yup_d$, *leads to* $fl_1(f, d) = ev \cdot \langle \pi_1 \cdot f, \ominus_d \rangle$ *Then, define function* $fl_2 : \mathcal{H}^2 X \to D$ *given as*

$$fl_2 = (+) \cdot \langle [\pi_2, i_2] \cdot ev_r, \pi_2 \rangle$$

$$\mathcal{H}^2 X \xrightarrow{\pi_2} D$$

$$\mathcal{H}^2 X \xrightarrow{ev_r} \mathcal{H}X + 1 \xrightarrow{[\pi_2, i_2]} D$$

Thus, $fl_2\,(f, d) = (\pi_2 \cdot f\ d \ \lhd (d \notin 1) \ \rhd \ i_2 \star) + d$. *Finally, we define for any* $X \in \mathbb{T}op$, $\mu_X = \langle fl_1, fl_2 \rangle$.

Intuitively, operation μ_X 'concatenates' functions; for example, given a pair (f, d), μ_X concatenates function $\pi_1 \cdot f$ - $0 : [0, d] \to X$ with $\pi_1 \cdot f d$ - $: [0, d'] \to X$, and sums the corresponding durations.

Lemma 2. *μ is a natural tranformation.*

$$\begin{tikzcd}
\mathcal{H}^2 X \arrow[r, "\mathcal{H}^2 h"] \arrow[d, "\mu_X"'] & \mathcal{H}^2 Y \arrow[d, "\mu_Y"] \\
\mathcal{H}X \arrow[r, "\mathcal{H}h"'] & \mathcal{H}Y
\end{tikzcd}$$

Proof. First we show that μ_X is continuous for any space $X \in \mathbb{T}op$. To do this, note that f, g are continuous and therefore

$$\cfrac{\cfrac{\cfrac{g : \mathcal{H}^2 X \times \mathbb{R}_0 \to X^{\mathbb{R}_0}}{\langle g, h \rangle : \mathcal{H}^2 X \times \mathbb{R}_0 \to X^{\mathbb{R}_0} \times \mathbb{R}_0}\ (\times)}{ev \cdot \langle g, h \rangle : \mathcal{H}^2 X \times \mathbb{R}_0 \to X}\ (\,\cdot\,)}{fl_1 : \mathcal{H}^2 X \to X^{\mathbb{R}_0}}\ (\,\lambda\,)$$

since fl_1, fl_2 are continuous we have

$$\dfrac{\dfrac{fl_1 : \mathcal{H}^2 X \to X^{\mathbb{R}_0}}{\langle fl_1, fl_2 \rangle : \mathcal{H}^2 X \to X^{\mathbb{R}_0} \times D}\ (\times)}{\langle fl_1, fl_2 \rangle : \mathcal{H}^2 X \to \mathcal{H} X}\ (\downarrow_r)$$

Hence, it remains to prove that the naturality square commutes.

$$\begin{aligned}
& \mathcal{H}h \cdot \mu_X = \mu_Y \cdot \mathcal{H}^2 h \\
\equiv \quad & \quad \{ \text{ Definition of } \mathcal{H}, \mu \ \} \\
& \big(h \cdot \times id\big) \cdot \langle fl_1, fl_2 \rangle = \langle fl_1, fl_2 \rangle \cdot \big(\mathcal{H}h \cdot \times id\big) \\
\equiv \quad & \quad \{ \text{ Absorption } \times \ \} \\
& \langle (h\cdot) \cdot fl_1, fl_2 \rangle = \langle fl_1, fl_2 \rangle \cdot \big(\mathcal{H}h \cdot \times id\big) \\
\equiv \quad & \quad \{ \text{ Equality } \times \ \} \\
& \begin{cases} (h\cdot) \cdot fl_1 = fl_1 \cdot \big(\mathcal{H}h \cdot \times id\big) \\ fl_2 = fl_2 \cdot \big(\mathcal{H}h \cdot \times id\big) \end{cases}
\end{aligned}$$

which leads to the cases below,

$$\begin{aligned}
& fl_1 \cdot \big(\mathcal{H}h \cdot \times id\big)\ (f, d) \\
= \quad & \quad \{ \text{ Application, definition of } fl_1 \ \} \\
& ev \cdot \langle \pi_1 \cdot \mathcal{H}h \cdot f, \ominus_d \rangle \\
= \quad & \quad \{ \text{ Natural } \pi_1 \ \} \\
& ev \cdot \langle (h\cdot) \cdot \pi_1 \cdot f, \ominus_d \rangle \\
= \quad & \quad \{ \text{ Definition of composition } \} \\
& h \cdot ev \cdot \langle \pi_1 \cdot f, \ominus_d \rangle \\
= \quad & \quad \{ \text{ Definition of } fl_1 \ \} \\
& (h\cdot) \cdot fl_1(f, d)
\end{aligned}$$

$$\begin{aligned}
& fl_2 \cdot (\mathcal{H}h \cdot \times id) \\
= \quad & \quad \{ \text{ Definition of } fl_2 \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r, \pi_2 \rangle \cdot (\mathcal{H}h \cdot \times id) \\
= \quad & \quad \{ \text{ Fusion } \times, \text{ natural } \pi_2 \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot (\mathcal{H}h \cdot \times id), \pi_2 \rangle
\end{aligned}$$

$$
\begin{aligned}
= & \quad \{ \text{ Definition of composition } \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot (\mathcal{H}h + id) \cdot ev_r, \pi_2 \rangle \\
= & \quad \{ \text{ Absorption } +\text{, definition of } \mathcal{H} \} \\
& (+) \cdot \langle [\pi_2 \cdot (h \cdot \times id), i_2] \cdot ev_r, \pi_2 \rangle \\
= & \quad \{ \text{ Natural } \pi_2 \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r, \pi_2 \rangle \\
= & \quad \{ \text{ Definition of } fl_2 \} \\
& fl_2
\end{aligned}
$$

□

Theorem 2. *The diagram below commutes.*

$$
\begin{tikzcd}
\mathcal{H} \arrow[r, "\eta_{\mathcal{H}}"] \arrow[dr, "1_{\mathcal{H}}"'] & \mathcal{H}^2 \arrow[d, "\mu"'] & \mathcal{H} \arrow[l, "\mathcal{H}\eta"'] \arrow[dl, "1_{\mathcal{H}}"] \\
& \mathcal{H} &
\end{tikzcd}
$$

Proof. Start with the triangle in the left

$$
\begin{aligned}
& id = \mu_X \cdot \eta_{\mathcal{H}X} \\
\equiv & \quad \{ \text{ Definition of } \mu \} \\
& id = \langle fl_1, fl_2 \rangle \cdot \eta_{\mathcal{H}X} \\
\equiv & \quad \{ \text{ Reflection } \times\text{, Equality } \times \} \\
& \begin{cases} \pi_1 = fl_1 \cdot \eta_{\mathcal{H}X} \\ \pi_2 = fl_2 \cdot \eta_{\mathcal{H}X} \end{cases}
\end{aligned}
$$

Unfolding both cases yields,

$$
\begin{aligned}
& fl_1 \cdot \eta_{\mathcal{H}X}\ (f, d) \\
= & \quad \{ \text{ Application } \} \\
& fl_1\ (\underline{(f, d)}, 0) \\
= & \quad \{ \text{ Definition of } fl_1 \} \\
& ev \cdot \langle \pi_1 \cdot \underline{(f, d)}, \ominus_0 \rangle \\
= & \quad \{ \text{ Definition of constant } \} \\
& f \cdot \ominus_0 \\
= & \quad \{ \ x - 0 = x \ \} \\
& f
\end{aligned}
$$

and then,

$$
\begin{aligned}
& fl_2 \cdot \eta_{\mathcal{H}X} \\
= & \quad \{ \text{ Definition of } fl_2 \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r , \pi_2 \rangle \cdot \eta_{\mathcal{H}X} \\
= & \quad \{ \text{ Fusion } \times, \text{ cancellation } \times \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \eta_{\mathcal{H}X} , i_1 \cdot \underline{0} \rangle \\
= & \quad \{ \ x + 0 = x \ \} \\
& [\pi_2, i_2] \cdot ev_r \cdot \eta_{\mathcal{H}X} \\
= & \quad \{ \ i_1 = ev_r \cdot \eta_{\mathcal{H}X} \ \} \\
& [\pi_2, i_2] \cdot i_1 \\
= & \quad \{ \text{ Cancellation } + \} \\
& \pi_2
\end{aligned}
$$

Now, consider the right triangle

$$
\begin{aligned}
& id = \mu_X \cdot \mathcal{H}\eta_X \\
\equiv & \quad \{ \text{ Definition of } \mu \ \} \\
& id = \langle fl_1, fl_2 \rangle \cdot \mathcal{H}\eta_X \\
\equiv & \quad \{ \text{ Reflection } \times, \text{ Equality } \times \ \} \\
& \begin{cases} \pi_1 = fl_1 \cdot \mathcal{H}\eta_X \\ \pi_2 = fl_2 \cdot \mathcal{H}\eta_X \end{cases}
\end{aligned}
$$

which leads to,

$$
\begin{aligned}
& fl_1 \cdot \mathcal{H}\eta_X \ (f, d) \\
= & \quad \{ \text{ Application } \} \\
& fl_1 \ (\eta_X \cdot f, d) \\
= & \quad \{ \text{ Definition of } fl_1 \ \} \\
& ev \cdot \langle \pi_1 \cdot \eta_X \cdot f, \ominus_d \rangle \\
= & \quad \{ \ \eta_X \cdot f = (\underline{f \text{ -}}, i_1 \cdot \underline{0}) \text{ , Cancellation } \times \ \} \\
& ev \cdot \langle \underline{f \text{ -}}, \ominus_d \rangle \\
= & \quad \{ \text{ Definition of constant } \} \\
& f
\end{aligned}
$$

$$
\begin{aligned}
& fl_2 \cdot \mathcal{H}\eta_X \\
= & \qquad \{ \text{ Definition of } fl_2, \text{ fusion } \times \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \mathcal{H}\eta_X, \pi_2 \cdot \mathcal{H}\eta_X \rangle \\
= & \qquad \{ \ ev_r \cdot \mathcal{H}\eta_X = (\eta_X + id) \cdot ev_r, \text{ natural } \pi_2 \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot (\eta_X + id) \cdot ev_r, \pi_2 \rangle \\
= & \qquad \{ \text{ Absorption } + \ \} \\
& (+) \cdot \langle [\pi_2 \cdot \eta_X, i_2] \cdot ev_r, \pi_2 \rangle \\
= & \qquad \{ \text{ Cancellation } + \ \} \\
& (+) \cdot \langle [i_1 \cdot \underline{0}, i_2] \cdot ev_r, \pi_2 \rangle \\
= & \qquad \{ \ 0 + x = x, i_2 \star = (i_2 \star) + (i_2 \star) \ \} \\
& \pi_2
\end{aligned}
$$

□

The following two lemmas are required to prove that $\langle \mathcal{H}, \eta, \mu \rangle$ is 'associative'.

Lemma 3. *The following equation holds*

$$ev \cdot \langle fl_1 \cdot f, \ominus_d \rangle = ev \cdot \langle \pi_1 \cdot ev \cdot \langle \pi_1 \cdot f, \ominus_d \rangle, \ominus_{d'} \cdot \ominus_d \rangle$$

Proof.

$$
\begin{aligned}
& ev \cdot \langle fl_1 \cdot f, \ominus_d \rangle \; x \\
= & \qquad \{ \text{ Application } \} \\
& fl_1 (f \; x) \; (x \ominus d) \\
= & \qquad \{ \text{ Application (let } \pi_1 \cdot f \; x = f', \pi_2 \cdot f \; x = d' \text{) } \} \\
& ev \cdot \langle \pi_1 \cdot f', \ominus_{d'} \rangle (x \ominus d) \\
= & \qquad \{ \text{ Application } \} \\
& \pi_1 \cdot f' \; (x \ominus d) \quad ((x \ominus d) \ominus d') \\
= & \qquad \{ \text{ Definition of } \ominus_{d'}, \ominus_d \ \} \\
& ev \cdot \langle \pi_1 \cdot f' \cdot \ominus_d, \ominus_{d'} \cdot \ominus_d \rangle \; x \\
= & \qquad \{ \text{ Definition of } f' \ \} \\
& ev \cdot \langle \pi_1 \cdot (\pi_1 \cdot f \; x) \cdot \ominus_d, \ominus_{d'} \cdot \ominus_d \rangle \; x \\
= & \qquad \{ \text{ Definition of product } \} \\
& ev \cdot \langle \pi_1 \cdot ev \cdot \langle \pi_1 \cdot f, \ominus_d \rangle, \ominus_{d'} \cdot \ominus_d \rangle \; x
\end{aligned}
$$

□

Lemma 4. *The following equation holds*

$$(+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \mu_{\mathcal{H}X}, [\pi_2, i_2] \cdot ev_r \rangle = [\pi_2, i_2] \cdot ev_r \cdot (\mu_X \cdot \times id)$$

Proof.

$$\begin{aligned}
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \mu_{\mathcal{H}X}, [\pi_2, i_2] \cdot ev_r \rangle \, (f, d) \\
= & \quad \{ \text{ Application (let } fl_2(f,d) = d_2) \ \} \\
& \pi_2\big(fl_1(f,d) \ \ d_2\big) + \pi_2(f \ d) \ \lhd \ (d \notin 1, d_2 \notin 1) \ \rhd \ i_2 \star \\
= & \quad \{ \text{ If } d_2 \notin 1 \text{ then } fl_1(f,d)\, d_2 = \pi_1 \cdot f \ d \ \ \pi_2(f \ d) \ \} \\
& \pi_2\big(\pi_1 \cdot f \ d \ \ \pi_2(f \ d)\big) + \pi_2(f \ d) \ \lhd \ (d \notin 1, d_2 \notin 1) \ \rhd \ i_2 \star \\
= & \quad \{ \ fl_2(f,d) \notin 1 \equiv d \notin 1, \pi_2(f \ d) \notin 1 \text{ (let } \pi_2(f \ d) = d') \ \} \\
& \big(\pi_2(\pi_1 \cdot f \ d \ \ d') + d' \ \lhd \ (d' \notin 1) \ \rhd \ i_2 \ \star\big) \ \lhd \ (d \notin 1) \ \rhd \ i_2 \star \\
= & \quad \{ \text{ Definition of } fl_2 \ \} \\
& fl_2 \cdot f \ d \ \lhd \ (d \notin 1) \ \rhd \ i_2 \star \\
= & \quad \{ \text{ Cancellation } \times \} \\
& \pi_2 \cdot \mu_X \cdot f \ d \ \lhd \ (d \notin 1) \ \rhd \ i_2 \star \\
= & \quad \{ \text{ Definitions of } ev_r \text{ and } [\pi_2, i_2] \ \} \\
& [\pi_2, i_2] \cdot ev_r \cdot (\mu_X \cdot \times id) \, (f, d)
\end{aligned}$$

□

Finally,

Theorem 3. *The diagram below commutes.*

$$\begin{tikzcd}
\mathcal{H}^3 \arrow[r, "\mu_{\mathcal{H}}"] \arrow[d, "\mathcal{H}\mu"'] & \mathcal{H}^2 \arrow[d, "\mu"] \\
\mathcal{H}^2 \arrow[r, "\mu"'] & \mathcal{H}
\end{tikzcd}$$

Proof. First,

$$\begin{aligned}
& \mu_X \cdot \mathcal{H}\mu_X = \mu_X \cdot \mu_{\mathcal{H}X} \\
\equiv & \quad \{ \text{ Definition of } \mu \ \} \\
& \langle fl_1, fl_2 \rangle \cdot \mathcal{H}\mu_X = \langle fl_1, fl_2 \rangle \cdot \mu_{\mathcal{H}X} \\
\equiv & \quad \{ \text{ Fusion } \times, \text{ Equality } \times \} \\
& \begin{cases} fl_1 \cdot \mathcal{H}\mu_X = fl_1 \cdot \mu_{\mathcal{H}X} \\ fl_2 \cdot \mathcal{H}\mu_X = fl_2 \cdot \mu_{\mathcal{H}X} \end{cases}
\end{aligned}$$

Then,

$$
\begin{aligned}
& fl_1 \cdot \mu_{\mathcal{H}X}(f,d) \\
= & \qquad \{ \text{ Definition of } fl_1 \ \ \} \\
& ev \cdot \langle \pi_1 \cdot (fl_1(f,d)), \ominus_e \rangle \\
= & \qquad \{ \text{ Definition of } fl_1 \ \ \} \\
& ev \cdot \langle \pi_1 \cdot ev \cdot \langle \pi_1 \cdot f, \ominus_d \rangle, \ominus_e \rangle \\
= & \qquad \{ \text{ Definition of } \ominus_e \ \} \\
& ev \cdot \langle \pi_1 \cdot ev \cdot \langle \pi_1 \cdot f, \ominus_d \rangle, \ominus_{d'} \cdot \ominus_d \rangle \\
= & \qquad \{ \ \text{ Lemma 3 } \ \} \\
& ev \cdot \langle fl_1 \cdot f, \ominus_d \rangle \\
= & \qquad \{ \text{ Cancellation } \times \ \} \\
& ev \cdot \langle \pi_1 \cdot \mu_X \cdot f, \ominus_d \rangle \\
= & \qquad \{ \ \text{ Definition of } fl_1 \ \ \} \\
& fl_1(\mu_X \cdot f, d) \\
= & \qquad \{ \text{ Definition of } \mathcal{H} \ \} \\
& fl_1 \cdot \mathcal{H}\mu_X \ (f,d)
\end{aligned}
$$

and

$$
\begin{aligned}
& fl_2 \cdot \mu_{\mathcal{H}X} \\
= & \qquad \{ \text{ Definition of } fl_2\text{, fusion } \times \ \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \mu_X, \pi_2 \cdot \mu_{\mathcal{H}X} \rangle \\
= & \qquad \{ \text{ Definition of } \mu_{\mathcal{H}X} \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \mu_{\mathcal{H}X}, fl_2 \rangle \\
= & \qquad \{ \text{ Definition of } fl_2 \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \mu_{\mathcal{H}X}, (+) \cdot \langle [\pi_2, i_2] \cdot ev_r, \pi_2 \rangle \rangle \\
= & \qquad \{ \text{ (+) is associative } \} \\
& (+) \cdot \langle (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot \mu_{\mathcal{H}X}, [\pi_2, i_2] \cdot ev_r \rangle, \pi_2 \rangle \\
= & \qquad \{ \ \text{ Lemma 4 } \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r \cdot (\mu_X \cdot \times id), \pi_2 \rangle \\
= & \qquad \{ \text{ Natural } \pi_2\text{, fusion } \times \ \} \\
& (+) \cdot \langle [\pi_2, i_2] \cdot ev_r, \pi_2 \rangle \cdot (\mu_X \cdot \times id) \\
= & \qquad \{ \text{ Definition } fl_2 \ \}
\end{aligned}
$$

$$
\begin{aligned}
& fl_2 \cdot (\mu_X \cdot \times id) \\
= & \qquad \{ \text{ Definition } \mathcal{H} \text{ } \} \\
& fl_2 \cdot \mathcal{H}\mu_X
\end{aligned}
$$

which concludes the proof. □

4. ... And its Kleisli category (Kl $\mathcal{H}$)

As mentioned above, the Kleisli category for $\mathcal{H}$ ($Kl\,\mathcal{H}$) provides the right setting to study the requirements placed by continuity over different forms of composition; actually, the envisaged component calculus is essentially its calculus. For future reference, let us recall its definition:

Definition 4. *Category $Kl\,\mathcal{H}$ is defined as follows*

- $|Kl\,\mathcal{H}| = |\mathbb{T}op|$,
- *for any objects $X, Y \in |Kl\,\mathcal{H}|$,*

$$Hom_{Kl\,\mathcal{H}}(X,Y) = Hom_{\mathbb{T}op}(X, \mathcal{H}Y)$$

- *for any space $X \in |Kl\,\mathcal{H}|$, η_X is its identity*
- *and, given two $Kl\mathcal{H}$ arrows $c_1 : X \to \mathcal{H}Y$, $c_2 : Y \to \mathcal{H}Z$ their (sequential) composition, denoted by $c_2 \bullet c_1$, is equal to*

$$\mu_Z \cdot \mathcal{H}c_2 \cdot c_1$$

In order to explore its structure, consider two arrows $c_1 : I \to \mathcal{H}K, c_2 : K \to \mathcal{H}O$. Then, denoting $\pi_1 \cdot c$ by f_c, for a continuous system $c : I \to \mathcal{H}O$, compute

$$
\begin{aligned}
& \pi_1 \cdot (c_2 \bullet c_1)\, x \\
= & \qquad \{ \text{ Definition of sequential composition } \} \\
& \pi_1 \cdot \mu_Z \cdot \mathcal{H}c_2 \cdot c_1\, x \\
= & \qquad \{ \text{ Cancellation } \times \text{ } \} \\
& fl_1 \cdot \mathcal{H}c_2 \cdot c_1\, x \\
= & \qquad \{ \text{ Definition of } \mathcal{H}\text{, let } \pi_2 \cdot c_1\, x = d \text{ } \} \\
& fl_1\, (c_2 \cdot (f_{c_1}\, x), d) \\
= & \qquad \{ \text{ Application } \} \\
& ev \cdot \langle \pi_1 \cdot c_2 \cdot (f_{c_1}\, x), \ominus_d \rangle \\
= & \qquad \{ \text{ Notation } \} \\
& ev \cdot \langle f_{c_2} \cdot (f_{c_1}\, x), \ominus_d \rangle
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& ev \cdot \langle f_{c_2} \cdot (f_{c_1}\, x), \ominus_d \rangle\, t \\
= \quad & \{ \text{ Application } \} \\
& f_{c_2}(f_{c_1}\, x\, t)\, (t \ominus d) \\
= \quad & \{ \ (f_{c_1}\, x) \cdot \curlywedge_d = f_{c_1}\, x \ \} \\
& f_{c_2}\, \big(\, f_{c_1}\, x\, (t \curlywedge d)\, \big)\, (t \ominus d)
\end{aligned}
$$

The last expression suggests a distinction between *passive* and *active* systems, which we formalise as follows. Consider a system $c : I \to \mathcal{H}I$; it leads to the arrow

$$init_c = I \xrightarrow{\ c\ } \mathcal{H}I \xrightarrow{\ \pi_1\ } I^{\mathbb{R}_0} \xrightarrow{ev \cdot \langle id, \underline{0} \rangle} I$$

We say that arrow c is passive if $init_c = id$ (*i.e.* $f_c\, x\, 0 = x$), active otherwise. The reason for this nomenclature will become clear in the following exercises.

Recall that

$$\pi_1 \cdot (c_2 \bullet c_1)\, x\, t = f_{c2}\, \big(f_{c1}\, x\, (t \curlywedge d)\, \big)\, (t \ominus d)$$

if c_2 is passive, the previous equation is equivalent to

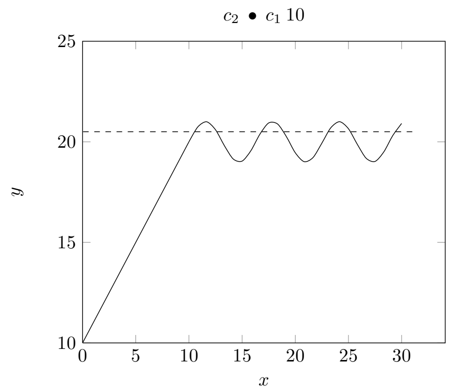
$$\pi_1 \cdot (c_2 \bullet c_1)\, x\, t = f_{c_1}\, x\, t \ \lhd \ (t \leq d) \ \rhd \ f_{c_2}(f_{c_1}\, x\, d)\, (t - d)$$

This means that for the duration of $f_{c_1}\, x$, $c_2 \bullet c_1\, x$ evolves according first to c_1, and then, on its termination, to c_2. Clearly, this is the expected behaviour according to the definition of operation μ which, as discussed in the previous section, 'concatenates' functions. The intuition about c_2 being active is also clear: until duration $[0, d]$ is completed, c_2 'alters' the evolution of c_1; then it proceeds according to its own evolution $f_{c_2}(f_{c_1}\, x\, d)$. This is illustrated in the following example.

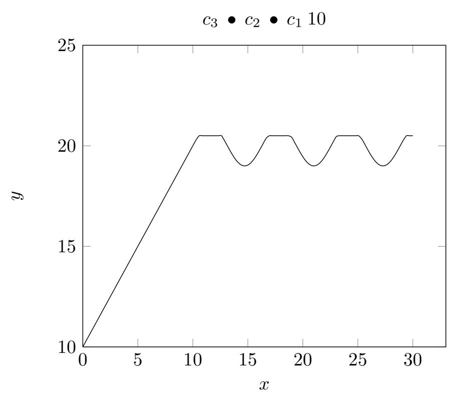
Example 1. *Suppose the temperature of a room is to be regulated according to the following discipline: starting at* $10\,°C$*, seek to reach and maintain* $20\,°C$*, but in no case surpass* $20.5\,°C$*. To realise such a system, three elementary components have to work together:* c_1 *to raise the temperature to* $20\,°C$*, component* c_2 *to maintain a given temperature, and component* c_3 *to ensure the temperature never goes over* $20.5\,°C$*. Formally,*

$$
\begin{aligned}
c_1\, x &= (\, (x + \ \text{-}\ ),\ 20 \ominus x\,) \\
c_2\, x &= (\, x + (\sin\ \ \text{-}\ ),\ \infty\,) \\
c_3\, x &= (\, \underline{x} \ \lhd \ (x \leq 20.5) \ \rhd \ \underline{20.5}\, , 0\,)
\end{aligned}
$$

In a first try one may compose c_2, c_1 *into* $c_2 \bullet c_1$*. This results in a component able to read the current temperature, raise it to* $20\,°C$*, and then keep it stable, as exemplified by the plot below.*



If, however, temperatures over $20.5\,^\circ C$ *occur, composition* $c_3 \bullet c_2 \bullet c_1$ *puts the system back into the right track as illustrated in the following plot.*



The example above hints at another interesting property.

Theorem 4. *Consider two arrows* $c_1 : I \to \mathcal{H}O$, $c_2 : O \to \mathcal{H}O$. *If* c_2 *is passive and* $\pi_2 \cdot c_1\ x = \star$, *then*

$$f_{(c_2 \bullet c_1)}\ x = f_{c_1}\ x$$

Proof.

$$
\begin{aligned}
& f_{(c_2 \bullet c_1)}\ x \\
= & \qquad \{\ \text{Notation}\ \} \\
& \pi_1 \cdot (c_2 \bullet c_1)\ x \\
= & \qquad \{\ \text{Definition of sequential composition}\ \} \\
& f_{c_2}(f_{c_1}\ x\ \text{-}\ )\ (\ \text{-}\ \ominus \infty) \\
= & \qquad \{\ \text{Definition of}\ \ominus\ \}
\end{aligned}
$$

$$
\begin{array}{ll}
& f_{c_2}(f_{c_1}\ x\ \text{-}\ )\ 0 \\
= & \qquad \{\ c_2 \text{ is passive }\ \} \\
& f_{c_1}\ x
\end{array}
$$

$\square$

Corollary 1. *If c_2 is passive and* $\operatorname{img} \pi_2 \cdot c_1 \subseteq 1$*, then* $c_2 \bullet c_1 = c_1$*.*

This means that if evolutions of the first component always exhibit an infinite duration, the second one, if passive, will never have the chance to execute.

(Co)limits are a main tool to build 'new' arrows from old, which in the case of $Kl\,\mathcal{H}$ translates to new forms of (continuous) component composition. Actually, coproducts are very easy to find through the canonical adjunction between $\mathbb{T}op$ and $Kl\,\mathcal{H}$,

$$
\begin{tikzcd}
\mathbb{T}op \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & Kl\,\mathcal{H} \arrow[l, bend left, "G"]
\end{tikzcd}
$$

which means that $Kl\,\mathcal{H}$ inherits the colimits of $\mathbb{T}op$ through F. One important colimit is the coproduct; in $Kl\,\mathcal{H}$ it is inherited as follows: given two components

$$
\begin{tikzcd}
X \arrow[dr, "c_1"'] & & Y \arrow[dl, "c_2"] \\
& Z &
\end{tikzcd}
$$

define component $[c_1, c_2] : X + Y \to \mathcal{H}Z$ such that

$$
\begin{tikzcd}
X \arrow[r, "\eta_{X+Y} \cdot i_1"] \arrow[dr, "c_1"'] & X + Y \arrow[d, "{[c_1, c_2]}"] & Y \arrow[l, "\eta_{X+Y} \cdot i_2"'] \arrow[dl, "c_2"] \\
& Z &
\end{tikzcd}
$$

i.e.

$$
\begin{aligned}
c_1 &= [c_1, c_2] \bullet (\eta_{X+Y} \cdot i_1) \\
c_2 &= [c_1, c_2] \bullet (\eta_{X+Y} \cdot i_2)
\end{aligned}
$$

Intuitively, $[c_1, c_2]$ behaves as c_1 whenever input X is chosen, and as c_2 otherwise.

Next, we explore the possibility of parallel evolutions. First note that the following pullback exists in $\mathbb{T}op$.

$$\begin{tikzcd}
I \arrow[rrd, bend left, "c_2"] \arrow[rd, dotted, "\langle c_1, c_2 \rangle"] \arrow[rdd, bend right, "c_1"'] & & \\
& \mathcal{H}K \times_{\mathcal{H}1} \mathcal{H}O \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{H}O \arrow[d, "\mathcal{H}!"] \\
& \mathcal{H}K \arrow[r, "\mathcal{H}!"'] & \mathcal{H}1
\end{tikzcd}$$

and that $\mathcal{H}K \times_{\mathcal{H}1} \mathcal{H}O \cong \mathcal{H}(K \times O)$. In particular,

$$\gamma = (i \times id) \cdot \langle \pi_1 \times \pi_1, \pi_2 \cdot \pi_2 \rangle$$

$$\mathcal{H}K \times_{\mathcal{H}1} \mathcal{H}O \xrightarrow{\langle \pi_1 \times \pi_1, \pi_2 \cdot \pi_2 \rangle} (K^{\mathbb{R}_0} \times O^{\mathbb{R}_0}) \times D \xrightarrow{i \times id} \mathcal{H}(K \times O)$$

where i is the iso

$$K^{\mathbb{R}_0} \times O^{\mathbb{R}_0} \cong (K \times O)^{\mathbb{R}_0}$$

Moreover,

Lemma 5. *The following equations hold*

$$\mathcal{H}\pi_1 \cdot \gamma = \pi_1$$
$$\mathcal{H}\pi_2 \cdot \gamma = \pi_2$$

Proof.

$$\begin{array}{ll}
& \mathcal{H}\pi_1 \cdot \gamma((f, d), (g, d)) \\
= & \qquad \{ \text{ Application } \} \\
& \mathcal{H}\pi_1(\langle f, g \rangle, d) \\
= & \qquad \{ \text{ Application } \} \\
& (\pi_1 \cdot \langle f, g \rangle, d) \\
= & \qquad \{ \text{ Cancellation } \times \} \\
& (f, d) \\
= & \qquad \{ \text{ Definition of } \pi_1 \} \\
& \pi_1((f, d), (g, d))
\end{array}$$

A similar proof establishes the other equation. ☐

Lemma 6. *Arrow* $\langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle : \mathcal{H}(K \times O) \to \mathcal{H}K \times \mathcal{H}O$ *is mono.*

Proof. Consider two arrows $f, g : I \to \mathcal{H}(K \times O)$ and assume that

$$\langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot f = \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot g$$

Then,

$$
\begin{array}{ll}
 & \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot f = \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot g \\
\Rightarrow & \qquad \{ \text{ Leibniz } \} \\
 & \gamma \cdot \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot f = \gamma \cdot \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot g \\
\equiv & \qquad \{ \ \gamma \cdot \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle = id \ \} \\
 & f = g
\end{array}
$$

□

Such intuitions lead to the following property,

Theorem 5. *The following pullback exists in* $Kl\,\mathcal{H}$.

$$
\begin{tikzcd}
I \arrow[dr, dotted, "\gamma \cdot \langle c_1, c_2 \rangle"] \arrow[drr, bend left, "c_2"] \arrow[ddr, bend right, "c_1"'] & & \\
 & K \times O \arrow[r, "\eta_O \cdot \pi_2"] \arrow[d, "\eta_K \cdot \pi_1"'] & O \arrow[d, "\eta_1 \cdot !"] \\
 & K \arrow[r, "\eta_1 \cdot !"'] & 1
\end{tikzcd}
$$

Proof. First, we show that the inner square commutes

$$
\begin{array}{ll}
 & (\eta_1 \cdot !) \bullet (\eta_K \cdot \pi_1) \\
= & \qquad \{ \text{ Definition of } \bullet \text{ and } \mathcal{H} \text{ is a functor } \} \\
 & \mu_1 \cdot \mathcal{H}\eta_1 \cdot \mathcal{H}! \cdot \eta_K \cdot \pi_1 \\
= & \qquad \{ \text{ Monadic laws } \} \\
 & \mathcal{H}! \cdot \eta_K \cdot \pi_1 \\
= & \qquad \{ \text{ Definition of } \langle \mathcal{H}, \eta, \mu \rangle \ \} \\
 & (! \cdot \times id) \cdot \langle (\_), \underline{0} \rangle \cdot \pi_1 \\
= & \qquad \{ \text{ Absorption } \times \ \} \\
 & \langle ! \cdot (\_), \underline{0} \rangle \cdot \pi_1 \\
= & \qquad \{ \text{ Definition of ! and constant } \ \} \\
 & \langle ! \cdot (\_), \underline{0} \rangle \cdot \pi_2 \\
= & \qquad \{ \text{ Absorption } \times \text{, definition of } \langle \mathcal{H}, \eta, \mu \rangle \ \} \\
 & \mathcal{H}! \cdot \eta_O \cdot \pi_2 \\
= & \qquad \{ \text{ Monadic laws } \}
\end{array}
$$

$$
\begin{aligned}
& \mu_1 \cdot \mathcal{H}\eta_1 \cdot \mathcal{H}! \cdot \eta_O \cdot \pi_2 \\
= & \qquad \{ \ \mathcal{H} \text{ is a functor, Definition of } \bullet \ \} \\
& (\eta_1 \cdot !) \bullet (\eta_O \cdot \pi_2)
\end{aligned}
$$

The left triangle commutes because (the proof for the right one is analogous):

$$
\begin{aligned}
& (\eta_K \cdot \pi_1) \bullet \gamma \cdot \langle c_1, c_2 \rangle \\
= & \qquad \{ \text{ Definition of } \bullet \text{ and } \mathcal{H} \text{ is a functor } \} \\
& \mu_K \cdot \mathcal{H}\eta_K \cdot \mathcal{H}\pi_1 \cdot \gamma \cdot \langle c_1, c_2 \rangle \\
= & \qquad \{ \text{ Monadic laws } \} \\
& \mathcal{H}\pi_1 \cdot \gamma \cdot \langle c_1, c_2 \rangle \\
= & \qquad \{ \text{ Lemma 5 } \} \\
& \pi_1 \cdot \langle c_1, c_2 \rangle \\
= & \qquad \{ \text{ Cancellation } \times \} \\
& c_1
\end{aligned}
$$

Finally, we need to prove uniqueness, *i.e.* that arrow $\gamma \cdot \langle c_1, c_2 \rangle$ is the only that makes the triangles commute. So assume there is an arrow f such that

$$
\begin{aligned}
& (\eta_K \cdot \pi_1) \bullet f = c_1 \wedge (\eta_K \cdot \pi_2) \bullet f = c_2 \\
\equiv & \qquad \{ \text{ Transitivity (twice) } \} \\
& (\eta_K \cdot \pi_1) \bullet f = (\eta_K \cdot \pi_1) \bullet \gamma \cdot \langle c_1, c_2 \rangle \wedge \\
& (\eta_O \cdot \pi_2) \bullet f = (\eta_O \cdot \pi_2) \bullet \gamma \cdot \langle c_1, c_2 \rangle \\
\equiv & \qquad \{ \text{ Definition of } \bullet \ (4\ \times), \mathcal{H} \text{ is a functor } (4\ \times) \ \} \\
& \mu_K \cdot \mathcal{H}\eta_K \cdot \mathcal{H}\pi_1 \cdot f = \mu_K \cdot \mathcal{H}\eta_K \cdot \mathcal{H}\pi_1 \cdot \gamma \cdot \langle c_1, c_2 \rangle \wedge \\
& \mu_O \cdot \mathcal{H}\eta_O \cdot \mathcal{H}\pi_2 \cdot f = \mu_O \cdot \mathcal{H}\eta_O \cdot \mathcal{H}\pi_2 \cdot \gamma \cdot \langle c_1, c_2 \rangle \\
\equiv & \qquad \{ \text{ Monadic laws (twice) } \} \\
& \mathcal{H}\pi_1 \cdot f = \mathcal{H}\pi_1 \cdot \gamma \cdot \langle c_1, c_2 \rangle \wedge \\
& \mathcal{H}\pi_2 \cdot f = \mathcal{H}\pi_2 \cdot \gamma \cdot \langle c_1, c_2 \rangle \\
\equiv & \qquad \{ \text{ Equality } \times, \text{ Fusion } \times \text{ (twice) } \} \\
& \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot f = \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \cdot \gamma \cdot \langle c_1, c_2 \rangle \\
\Rightarrow & \qquad \{ \ \langle \mathcal{H}\pi_1, \mathcal{H}\pi_2 \rangle \text{ is mono (Lemma 6) } \} \\
& f = \gamma \cdot \langle c_1, c_2 \rangle
\end{aligned}
$$

□

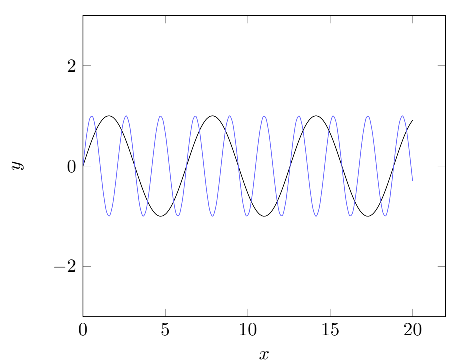
The theorem asserts the existence of products whenever, for the same input, the duration of evolutions coincide. It also says that whenever two components are compatible – in the sense that similar inputs always produce evolutions with equal duration – a new component can be defined representing their parallel composition. This actually brings parallelism up front, and, moreover, makes possible to combine evolutions, as illustrated in the following example.

Example 2. *Consider two signal generators, c_1, c_2 such that*

$$\begin{aligned} c_1\ x &= (\ x + (\sin\ \ \text{-}\ ), 20\ ) \\ c_2\ x &= (\ x + \sin\,(3 * \ \text{-}\ ), 20\ ) \end{aligned}$$

For input 0*, their parallel evolution* $\gamma \cdot \langle c_1, c_2 \rangle\ 0$ *(denoted in the sequel by* $\langle c_1, c_2 \rangle^{\mathcal{H}}\ 0$*) is as follows*

$\langle c_1, c_2 \rangle^{\mathcal{H}}\ 0$

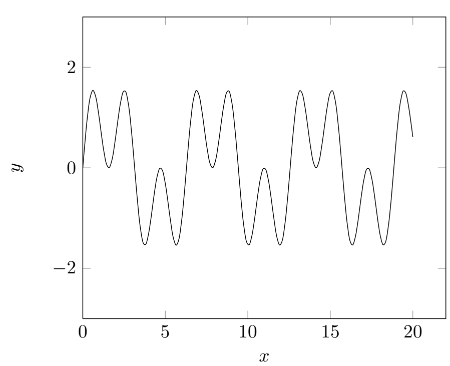


Moreover, we can combine signals. For example, to add incoming signals, take the active component c_3*, formally defined as*

$$c_3(x, y) = (\underline{x + y}, 0)$$

For input 0*, the system* $c_3 \bullet \langle c_1, c_2 \rangle^{\mathcal{H}}$ *yields the plot below*

$c_3 \bullet \langle c_1, c_2 \rangle^{\mathcal{H}}\ 0$



We close this section further investigating the structure of monad $\mathcal{H}$. We start by defining a tensorial strength for monad $\mathcal{H}$ — which turns out to be an essential mechanism for the generation of a calculus for hybrid components.

Definition 5. *Define the function* $f_1 : \mathcal{H}X \times Y \to (X \times Y)^{\mathbb{R}_0}$ *such that*

$$f_1 = i \cdot (\pi_1 \times \lambda\pi_1)$$
$$\mathcal{H}X \times Y \xrightarrow{(\pi_1 \times \lambda\pi_1)} X^{\mathbb{R}_0} \times Y^{\mathbb{R}_0} \xrightarrow{\quad i \quad} (X \times Y)^{\mathbb{R}_0}$$

and function $f_2 : \mathcal{H}X \times Y \to D$ *such that* $f_2 = \pi_2 \cdot \pi_1$*. Then, denote function* $\langle f_1, f_2 \rangle : \mathcal{H}X \times Y \to \mathcal{H}(X \times Y)$ *by symbol* τ_r*.*

Theorem 6. $\mathcal{H}$ *is strong.*

Proof. We will show that τ_r corresponds to the uniform characterisation of right tensorial strength (which entails that $\mathcal{H}$ is strong).

$$\begin{aligned}
& \tau_r\big((f, d), y\big) \\
= & \quad \{ \text{ Application } \} \\
& (\langle f, \underline{y} \rangle, d) \\
= & \quad \{ \text{ Definition of constant } \} \\
& ((_, y) \cdot f, d) \\
= & \quad \{ \text{ Definition of } \mathcal{H} \ \} \\
& \mathcal{H}(_, y)\,(f, d)
\end{aligned}$$

□

The monad $\mathcal{H}$, however, fails commutativity, *i.e.* equation

$$\tau_r \bullet \tau_l = \tau_l \bullet \tau_r$$

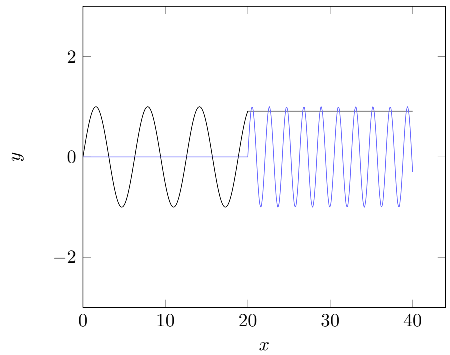
does not hold, as shown in the following counter–example.

Example 3. *Recall Example 2; it introduced two signal generators, formally defined as*

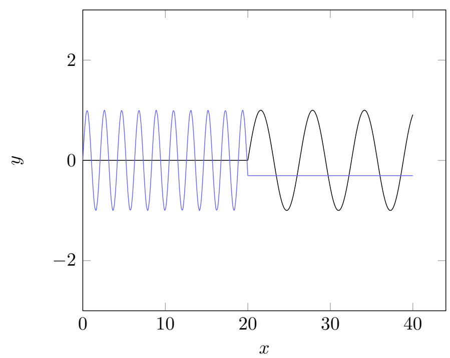
$$\begin{aligned}
c_1\ x\ &= (\ x + (\sin\ \ \text{-}\ ), 20\ ) \\
c_2\ x\ &= (\ x + \sin\,(3 *\ \ \text{-}\ ), 20\ )
\end{aligned}$$

Thus, we have

$\tau_r \bullet \tau_l \cdot \langle c_1, c_2 \rangle\, 0$



$\tau_l \bullet \tau_r \cdot \langle c_1, c_2 \rangle\, 0$



Indeed, the example above shows that $\tau_r \bullet \tau_l \neq \tau_l \bullet \tau_r$. Nonetheless, the plots illustrate an interesting behaviour: $\tau_r \bullet \tau_l \cdot \langle c_1, c_2 \rangle$ means "first let the component in the left to act, then the one in the right"; and conversely for $\tau_l \bullet \tau_r \cdot \langle c_1, c_2 \rangle$. Moreover, note that each component 'waits' for the other by stalling its evolution, which seems to introduce a mechanism for synchronisation.

5. Conclusions and future work

Software systems are becoming prevalently intertwined with (continuous) physical processes. This renders their rigorous design (and analysis) a difficult challenge that calls for a wide, uniform framework where 'Continuous' Mathematics and Computer Science must work together.

As a first step towards a calculus of hybrid components, in the spirit of [2], this paper showed how continuous behaviour can be encoded in the form of a strong topological monad. The paper further explored the corresponding Kleisli category where the effects of continuity over different forms of composition can be isolated and studied.

Related work. During the last two decades, a few categorial models for hybrid systems were proposed. For example, document [14] introduced an institution – in essence, a categorial rendering of logic – for hybrid systems and provided basic forms of composition such as free aggregation (*i.e.* parallelism without interaction) and interconnection where some attributes and events are shared between two systems. Around the same time, Jacobs [15] suggested a coalgebraic framework where hybrid systems are viewed as coalgebras equipped with a monoid action: coalgebras define the discrete transitions, and monoid actions the continuous evolutions. Some years later Haghverdi *et. al* [16] provided an interesting formalisation of hybrid systems using a conceptual framework which is closer to the coalgebraic perspective. In fact, their objective in the paper was to give appropriate notions of bisimulation both for dynamical and hybrid systems. Composition mechanisms, however, were not studied.

The monad introduced in this paper captures the typical continuous behaviour of hybrid systems. Actually, there is a close relationship between the work reported here and Peter Höfner's algebra of hybrid systems [17]: the latter's main operator and its laws are embedded in the (sequential) composition of $Kl\,\mathcal{H}$, in particular when restricted to passive systems. Moreover, the algebra possesses secondary operators, used for synchronisation purposes, that can also be found in the structure of $Kl\,\mathcal{H}$. As already mentioned, our approach, and differently from Höfner's calculus, is structured around a monad that encodes continuous evolution; this brings a number of canonical constructions and smooths the integration with other behavioural effects, such as non determinism or probabilistic evolution.

Future work. Our current research investigates how hybrid behaviour can be rendered by arrows typed as $\langle c, p\rangle : S \times I \to S \times \mathcal{H}O$, where $c : S \times I \to S$ is a discrete arrow (S comes equipped with the discrete topology) and $p : S \times I \to \mathcal{H}O$ is a continuous system. This paves the way to extending the component calculus in [2] to hybrid systems.

A second research line concerns the development of a taxonomy of continuous systems 'living' in $Kl\mathcal{H}$. Topologies can be useful to elicit a number of important properties of hybrid systems, as Stauner showed more than a decade ago [18]. For instance, the notion of stability, essential in control theory, is very simple to formulate in $Kl\,\mathcal{H}$: a system is called stable when small input variations originate small variations in the output. In $Kl\,\mathcal{H}$ this essentially means that the input space is not equipped with the discrete topology.

Acknowledgements This work is funded by ERDF - European Regional Development Fund, through the COMPETE Programme, and by National Funds through FCT within project FCOMP-01-0124-FEDER-028923. The first author is also sponsored by FCT grant SFRH/BD/52234/2013.

References

[1] P. Tabuada, Verification and Control of Hybrid Systems - A Symbolic Approach, Springer, 2009.

[2] L. S. Barbosa, Towards a calculus of state-based software components, Journal of Universal Computer Science 9 (2003) 891–909.

[3] A. Kock, Strong functors and monoidal monads, Archiv der Mathematik 23 (1) (1972) 113–120.

[4] E. Moggi, Notions of computation and monads, Information and computation 93 (1) (1991) 55–92.

[5] P. Wadler, Monads for functional programming, in: Advanced Functional Programming, Springer, 1995, pp. 24–52.

[6] S. Marlow, R. Newton, S. L. P. Jones, A monad for deterministic parallelism, in: Proceedings of the 4th ACM SIGPLAN Symposium on Haskell, Haskell 2011, Tokyo, Japan, 22 September 2011, 2011, pp. 71–82. doi:10.1145/2034675.2034685.
URL http://doi.acm.org/10.1145/2034675.2034685

[7] E.-E. Doberkat, Stochastic Coalgebraic Logic, Monographs in Theoretical Computer Science. An EATCS Series, Springer, 2009.

[8] I. Hasuo, B. Jacobs, A. Sokolova, Generic trace theory, Electr. Notes Theor. Comput. Sci. 164 (1) (2006) 47–65. doi:10.1016/j.entcs.2006.06.004.
URL http://dx.doi.org/10.1016/j.entcs.2006.06.004

[9] E. Moggi, Computational lambda-calculus and monads, in: Proceedings of the Logic in Computer Science Conference, 1989.

[10] L. S. Barbosa, J. N. Oliveira, Transposing partial components: an exercise on coalgebraic refinement, Theor. Comp. Sci. 365 (1-2) (2006) 2–22.

[11] J. N. Oliveira, Preparing relational algebra for "just good enough" hardware, in: P. Höfner, P. Jipsen, W. Kahl, M. E. Müller (Eds.), Relational and Algebraic Methods in Computer Science - 14th International Conference, RAMiCS 2014, Marienstatt, Germany, April 28-May 1, 2014. Proceedings, Vol. 8428 of Lecture Notes in Computer Science, Springer, 2014, pp. 119–138. doi:10.1007/978-3-319-06251-8_8.
URL http://dx.doi.org/10.1007/978-3-319-06251-8_8

[12] R. Bird, O. de Moor, The Algebra of Programming, Prentice-Hall, 1996.
URL http://www.cs.ox.ac.uk/publications/books/algebra/

[13] M. Escardó, R. Heckmann, Topologies on spaces of continuous functions, in: Topology Proceedings, Vol. 26, 2001, pp. 545–564.

[14] H. Lourenço, A. Sernadas, An institution of hybrid systems, in: D. Bert, C. Choppy, P. Mosses (Eds.), Recent Trends in Algebraic Development Techniques, Vol. 1827 of Lecture Notes in Computer Science, Springer Berlin Heidelberg, 2000, pp. 219–236. doi:10.1007/978-3-540-44616-3_13.
URL http://dx.doi.org/10.1007/978-3-540-44616-3_13

[15] B. Jacobs, Object-oriented hybrid systems of coalgebras plus monoid actions, Theoretical Computer Science 239 (1) (2000) 41 – 95. doi:http://dx.doi.org/10.1016/S0304-3975(99)00213-3.
URL http://www.sciencedirect.com/science/article/pii/S0304397599002133

[16] E. Haghverdi, P. Tabuada, G. J. Pappas, Bisimulation relations for dynamical, control, and hybrid systems, Theor. Comput. Sci. 342 (2-3) (2005) 229–261. doi:10.1016/j.tcs.2005.03.045.
URL http://dx.doi.org/10.1016/j.tcs.2005.03.045

[17] P. Höfner, Algebraic calculi for hybrid systems, Ph.D. thesis, University of Augsburg (2009).

[18] T. Stauner, Systematic development of hybrid systems, Ph.D. thesis, TU München (2001).