SEMANTICS OF NON-TERMINATING REWRITE SYSTEMS USING MINIMAL COVERINGS

by

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Technical Monograph PRG-118

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Abstract

We propose a new semantics for rewrite systems based on interpreting rewrite rules as inequations between terms in an ordered algebra. In particular, we show that the algebra of normal forms in a terminating system is a uniqnely minimal covering of the term algebra. In the non-terminating case, the existence of this minimal covering is established in the completion of an ordered algebra formed by rewriting sequences. We thus generalize the properties of normal forms for non-terminating systems to this minimal covering. These include the existence of normal forms for arbitrary rewrite systems, and their uniqueness for confluent systems, in which case the algebra of normal forms is isomorphic to the canonical quotient algebra associated with the rules when seen as eqnations. This extends the benefits of algebraic semantics to systems with non-deterministic and non-terminating computations. We first study properties of abstract orders, and then instantiate these to term rewriting systems.

1 Introduction

Term rewriting is the the basic computational aspect of equational logic and is fnndamental to prototyping algebraic specifications. The vast majority of the literature in this area focnses on terminating rewrite systems, i.e., systems where no infinite rewriting sequence occurs But there is now increasing research on the semantics of non-terminating systems. Non-strict functional languages such as MIRANDA [23] provide a practical reason to study such systems, since one can write non-terminating functions that "compute" infinite structures, such as the list of all prime numbers. Moreover, it is often desirable to write terminating functions using other functions whose termination cannot be established. Such use of intermediate non-terminating fnnctions may seem less peculiar if we note an analogous technique in imperative languages: it is not unusual to see in a terminating C program a loop of the form

*Also Universidade do Minho, Braga, Portugal. Supported in part by JNICT under contract BD-1102-90/IA.

†The research reported in this paper has been supported in part by the Science and Engineering Research Council, the CEC under ESPRIT-2 BRA Working Groups 6071, IS-CORE (Information Systems COrrectness and REusability) and 6112, COMPASS (COMPrehensive Algebraic Approach to System Specification and development), Fujitsu Laboratories Limited, and under the management of the Information Technology Promotion Agency (IPA), Japan, as part of the Industrial Science and Technology Frontier Program "New Models for Software Architectures," sponsored by NEDO (New Energy and Industrial Technology Development Organization).

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while (1) { ... }
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Another application is in specifying reactive or stream-based programs: for example, an operating system should not terminate. Also, studying non-terminating rewrite systems deepens our knowledge of rewrite systems in general. More specifically, it shows us which properties of a terminating rewrite system arise just because it terminates, and which are independent from termination.

The usual semantics for rewrite system is based in interpreting rewrite rules as equations and rewriting as a particular case of equational reasoning. Our proposal is different. Rules have a computational meaning – rewriting a term is computing it, i.e., finding its **value**, which is just a non-reducible term (a normal form). The connection with equations is seen in another way – in certain cases, we can replace equational reasoning by another type of reasoning: two terms are equal if they have the same value, i.e., if the result of computing them is equal. This can only be done if we guarantee that every term has a unique value and every term is equal to its value. The termination of a rewrite system ensures that every term has a value (normal form). But, in general we cannot guarantee this.

The research that has been done on non-terminating rewrite systems [21, 4, 6, 17, 20, 22, 7, 5, 16] is centered on seeking semantics for these systems where the usual properties of confluent systems (like uniqueness of normal forms) still hold. Most research that has been done on the semantics of non-terminating rewrite systems follows ideas of the ADJ Group on continuous algebras [12], where the authors give an elegant algebraic definition of finite and infinite terms as a way of completing the term algebra: they show not only that the algebra of finite and infinite terms is continuous but also that every infinite term is the least upper bound of a set of finite terms. These approaches extend the original set of terms (with infinite terms) in such a way that every term has a value. The problem with these approaches is that the connection referred above between rewriting and equational reasoning is not preserved: terms that are not equal can have the same value! Kennaway *et al.* [16] show that even in confluent systems, the ω-normal forms defined in [7] are not unique. Also the existence of these ω-normal forms, as well as infinite normal forms [16] is not only dependent on the confluence of a rewrite system, but on other properties like left-linearity and top-termination.

Our answer to this problem is to interpret rewrite rules as **inequations**. We then have a variant of equational logic (the logic of replacing equals for equals) called **inequational logic** – the logic of replacing terms by larger terms[1]. The models in this logic are **preordered algebras** - algebras whose carrier is a preordered set. The term algebra is now a preorder – a term t is above a term t' iff all values of t are values of t' – and the set of values is nothing more than the maximal elements in this preorder. This view of rewriting is somehow similar to the algebraic definition of refinement (e.g. [14]). This view is also consistent with the work of Meseguer [19, 18], where it is argued that rules express change in a computational system. The main difference from inequational logic and Meseguer's **rewriting logic** is that, apart from limiting ourselves to the unconditional case, we do not record in any way how a reduction was performed. In rewriting logic, each reduction $t \rightarrow^*_{\mathcal{R}} t'$ is associated with the sequence of (parallel) rule applications. Since we omit this information, we cannot distinguish between two different reductions with the same start and end points. However, the simplicity of our approach makes it quite elegant where it applies, as demonstrated by the proofs of completeness and soundness of inequational logic given in Section 4.

[1] In standard rewriting texts (e.g., [15]) rewriting is often associated with simplification; thus we should have said *replacing terms by smaller terms*. The reason for using our terminology comes from the fact that in a great part of the examples of non-terminating rewrite systems, rewriting increases the size of terms.

In the case of terminating and confluent rewrite systems, normal forms constitute an initial algebra. We show that this algebra has a very special property: it is a **uniquely minimal covering** of the term algebra. It is this property that makes it the obvious choice for implementing the abstract data type described by the rules, and moreover, its initiality is provable from just that property. We also show that for globally finite rewrite systems, the results proved by Goguen [8] follow from the existence of a uniquely minimal covering of the term algebra.

To deal with arbitrary non-terminating systems, instead of extending the preorder of terms, we use another preorder – of the rewriting sequences – together with an injective embedding from the original preorder into this other one. This preorder has the important property mentioned above about the preorder of terms in the terminating case: all elements have a value, that we call a *normalizing sequence*. In other words, this preorder has a minimal covering. In the case of confluent systems these values are unique, allowing us to generalize the properties of the algebra of normal forms to this one. Among the differences between the results obtained here and in the cited approaches, we would emphasize the existence of normal forms for arbitrary systems, the uniqueness of these normal forms in confluent systems, and in this last case, the isomorphism between these normal forms and the canonical quotient algebra. We feel that the success of the approach presented here paves the way to applications of rewrite systems to concurrency, e.g., results along the lines of Hennessy [13] and Meseguer [19, 18].

Section 2 introduces the notation that we will use. In Section 3 we present some abstract properties of complete preorders and minimal coverings. Finally, we apply these properties to the particular case of term rewriting systems in Section 4. For simplicity of exposition we present here only the unsorted case, but everything extends smoothly to the many-sorted case.

2 Preliminaries

A **preorder** $(X, \sqsubseteq_X)$ consists of a set X and a reflexive and transitive binary relation $\sqsubseteq_X$ over X. A preorder is a **partial order** iff $\sqsubseteq_X$ is anti-symmetric; it is an **equivalence** iff $\sqsubseteq_X$ is symmetric. Let $(A, \equiv)$ be an equivalence; for each $a \in A$, let $[a]_\equiv$ denote the set $\{b \in A \mid a \equiv b\}$ and let $A/_\equiv$ denote the set $\{[a]_\equiv \mid a \in A\}$. Given preorders $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$, a mapping $f : X \to Y$ is **monotonic** iff $f(x) \sqsubseteq_Y f(y)$ whenever $x \sqsubseteq_X y$. It is an **order embedding** iff for every $x, y \in X$, $x \sqsubseteq_X y$ iff $f(x) \sqsubseteq_Y f(y)$. Given a subset X of A, an upper bound of X is an element $a \in A$ such that $\forall x \in X\ x \sqsubseteq a$. An upper bound a of X is a **least upper bound** (**lub**) iff for any upper bound a' of X, $a \sqsubseteq a'$.

Given a preorder A, a non-empty subset C of A is a **chain** iff it is totally ordered (i.e., for every $x_1, x_2 \in C$ either $x_1 \sqsubseteq x_2$ or $x_2 \sqsubseteq x_1$). A chain is an **ω-chain** iff it is denumerable. A non-empty subset Δ of A is **directed** iff for every pair of elements d_1 and d_2 of Δ there exists an element d in Δ such that both $d_1 \sqsubseteq_A d$ and $d_2 \sqsubseteq_A d$. A preorder A is **ω-complete** iff every ω-chain has a lub in A, and is **complete** iff every directed subset of A has a lub in A. If $\sqsubseteq$ is a partial order then the lub of any set Δ, if it exists, is unique and is denoted by $\bigsqcup \Delta$. A monotonic mapping $f : A \to B$ between ω-complete preorders is **ω-continuous** iff it preserves least upper bounds of ω-chains, i.e., for every ω-chain X, if x is a lub of X then $f(x)$ is a lub of $f(X)$. Similarly, if A and B are complete preorders, f is **continuous** iff for every directed subset X of A, if x is a lub of X then $f(x)$ is a lub of $f(X)$.

Given a preorder A we define its kernel, denoted $\simeq$, as the largest equivalence contained in it, i.e., for all x in A,

$$x \simeq y \ \text{ iff } \ x \sqsubseteq y \ \text{ and } \ y \sqsubseteq x$$

We also define the partial order $A/_{\simeq} = (A/_{\simeq}, \sqsubseteq_{\simeq})$ by

$$[x]_{\simeq} \sqsubseteq_{\simeq} [y]_{\simeq} \text{ iff } x \sqsubseteq y$$

For each monotonic mapping $f : A \to B$, we define the monotonic mapping $f_{\simeq} : A/_{\simeq} \to B/_{\simeq}$ to send each $[a]_{\simeq}$ to $[f(a)]_{\simeq}$.

Given a preorder $A = (A, \sqsubseteq)$, we define $(A_\omega, \ll)$ to be the preorder where A_ω is the set of ω-chains of elements of A, and

$$a \ll b \text{ iff } \forall i\ \exists j\ a_i \sqsubseteq b_j$$

For each monotonic mapping $f : A \to B$, we define the mapping $f_\omega : A_\omega \to B_\omega$ by

$$f_\omega(\langle a_0, a_1, \ldots\rangle) = \langle f(a_0), f(a_1), \ldots\rangle$$

It is well known [3] that A_ω is an ω-complete preorder, and that f_ω is ω-continnous. Moreover,

Proposition 2.1 ([1]) *If A is denumerable then A_ω is complete and f_ω is continuous.*

Given a preorder $(A, \sqsubseteq_A)$, let $\equiv_A$ be defined as $a \equiv_A b$ if there exists a sequence $\langle a_0, \ldots a_n\rangle$ of elements of A such that $a = a_0$, $b = a_n$, and for each $0 \leqslant i < n$ $a_i \sqsubseteq_A a_{i+1}$ or $a_{i+1} \sqsubseteq_A a_i$. In other words, $\equiv_A$ is the symmetric and transitive closure of $\sqsubseteq_A$. We denote by $A_{\equiv}$ the set $\{[a]_{\equiv_A} \mid a \in A\}$. Given a monotonic mapping $f : A \to B$, $f_{\equiv} : A_{\equiv} \to B_{\equiv}$ sends each $[a]_{\equiv_A}$ to $[f(a)]_{\equiv_B}$.

2.1 Algebras and Equations

A **signature** Σ is a family $\Sigma = \{\Sigma_n\}_{n\in\omega}$. An element $\sigma \in \Sigma_n$ is called a **function symbol** of **arity** n, and in particular, an element of Σ_0 is called a **constant symbol**. A signature Σ where $\Sigma_n = \varnothing$ for all $n > 0$ is called a **ground signature**, and is basically just a set of symbols. Given signatures Σ and Ω their union $\Sigma \cup \Omega$ is defined as $(\Sigma\cup\Omega)_n = \Sigma_n \cup \Omega_n$; Σ and Ω are said to be **disjoint** if $\bigcup_n \Sigma_n$ and $\bigcup_n \Omega_n$ are disjoint.

The set T_Σ of all Σ-terms is the smallest set of *strings* over $(\bigcup_n \Sigma_n) \cup \{\underline{(},\underline{)},\underline{,}\}$ (where $\underline{(}$, $\underline{)}$, and $\underline{,}$ are special symbols disjoint from Σ) that contains Σ_0 and such that $\sigma\underline{(}t_1\underline{,}\ldots\underline{,}t_n\underline{)} \in T_\Sigma$ whenever each $t_i \in T_\Sigma$. We will often omit the underlying of these symbols. For a ground signature X, we denote by $T_\Sigma(X)$ the Σ-algebra $T_{\Sigma\cup X}$.

A **Σ-algebra** is a set A together with a function $A_\sigma : A^n \to A$ for each $\sigma \in \Sigma_n$. In particular, if $n = 0$, A_σ is just an element of A. A **Σ-homomorphism** between Σ-algebras A and B is a mapping $h : A \to B$ such that $h(A_\sigma(a_1, \ldots, a_n)) = B_\sigma(h(a_1), \ldots, h(a_n))$ for every $\sigma \in \Sigma_n$.

T_Σ can be seen a Σ-algebra in the obvious way. A key property of this Σ-algebra is **initiality**:

Theorem 2.2 *For any Σ-algebra A, there exists a* **unique** *Σ-homomorphism from T_Σ to A.*

Corollary 2.3 *For any ground signature X disjoint from Σ, Σ-algebra A, and mapping $\theta : X \to A$ (such a mapping is often called an* **assignment***), there exists a unique Σ-homomorphism $\overline{\theta} : T_\Sigma(X) \to A$ that extends θ in the sense that $\overline{\theta}(x) = \theta(x)$.*

In the particular case where A is $T_\Sigma(Y)$ then an assignment θ is often referred as a **substitution** and $\overline{\theta}$ is the mapping that applies the substitution θ to terms.

Given a signature Σ, a **Σ-equation** (or equation if the signature is understood from the context) is a triple (X, l, r) where X is a set of variables (i.e., a ground signature) disjoint from Σ, and l and r are Σ-terms. We often write an equation in the form $(\forall X)\ l = r$. A Σ-algebra A **satisfies** the equation $(\forall X)\ l = r$ if for all assignments $\theta : X \to A$, $\overline{\theta}(l) = \overline{\theta}(r)$. A Σ-algebra A satisfies a set E of equations if it satisfies each of the equations in that set.

Given a set E of Σ-equations, $(T_\Sigma, \equiv_E)$ is the least equivalence that such that

- for all equations $(\forall X)\ l = r$ in E and assignments $\theta : X \to T_\Sigma$, $\overline{\theta}(l) \equiv_E \overline{\theta}(r)$,
- for each $\sigma \in \Sigma_n$, $\sigma(t_1, \ldots, t_n) \equiv \sigma(t'_1, \ldots, t'_n)$ whenever for all $1 \leqslant i \leqslant n$, $t_i \equiv_E t'_i$.

We can make $T_\Sigma/_{\equiv_E}$ into a Σ-algebra:

- for each $\sigma \in \Sigma_0$, $(T_\Sigma/_{\equiv_E})_\sigma = [\sigma]_{\equiv_E}$,
- for each $\sigma \in \Sigma_n$, $(T_\Sigma/_{\equiv_E})_\sigma([t_1]_{\equiv_E}, \ldots, [t_n]_{\equiv_E}) = [\sigma(t_1, \ldots, t_n)]_{\equiv_E}$

Again this algebra has an important property:

Theorem 2.4 *For any Σ-algebra A that satisfies the equations in E, there exists a* **unique** *Σ-homomorphism from $T_\Sigma/_{\equiv_E}$ to A.*

2.2 Preordered Algebras and Inequations

A **preordered** (resp. **partially ordered**) **Σ-algebra** is a preorder (resp. partial order) $(A, \sqsubseteq_A)$ called the carrier of the algebra, together with a monotonic mapping $A_\sigma : A^n \to A$ for each $\sigma \in \Sigma_n$. A preordered Σ-algebra is **continuous** (resp. **ω-continuous**) if its carrier is complete (resp. ω-complete) and each of the functions is continuous (resp. ω-continuous).

Given a preordered algebra A we define:

- the partially ordered Σ-algebra $A/_\simeq$ as having carrier $(A/_\simeq, \sqsubseteq_\simeq)$, and for each $\sigma \in \Sigma$, $(A/_\simeq)_\sigma = (A_\sigma)_\simeq$;
- the ω-continuous preordered Σ-algebra A_ω as having carrier $(A_\omega, \ll)$, and for each $\sigma \in \Sigma$, $(A_\omega)_\sigma = (A_\sigma)_\omega$;
- the Σ-algebra $A_\equiv$ as having carrier $A_\equiv$ and for each $\sigma \in \Sigma$, $(A_\equiv)_\sigma = (A_\sigma)_\equiv$.

Given a signature Σ, a **Σ-inequation** (or just inequation if the signature is understood from the context) is a triple (X, l, r) where X is a set of variables disjoint from Σ and l and r are Σ-terms. We often write an inequation in the form $(\forall X)\ l \sqsubseteq r$. A preordered Σ-algebra A **satisfies** the inequation $(\forall X)\ l \sqsubseteq r$ if for all assignments $\theta : X \to A$, $\overline{\theta}(l) \sqsubseteq_A \overline{\theta}(r)$; A satisfies a set $\mathcal{R}$ of inequations if it satisfies each of the inequations in that set.

Theorem 2.5 *If A is a preordered Σ-algebra that satisfies a set $\mathcal{R}$ of inequations then so do $A/_\simeq$ and A_ω (and thus $A_\omega/_\simeq$).*

The relation between equations and inequations is expressed by the following:

Theorem 2.6 *If A is a preordered Σ-algebra that satisfies the inequation $(\forall X)\ l \sqsubseteq r$ then $A_\equiv$ satisfies the equation $(\forall X)\ l = r$.*

The implication is proper and truly characterizes the relationship between equations and inequations.

3 Complete Orders and Minimal Coverings

A preorder $(A, \sqsubseteq)$ is **terminating** if there exists no infinite chain

$$a_0 \sqsubset a_1 \sqsubset \cdots \sqsubset a_n \sqsubset \cdots$$

where $a \sqsubset a'$ iff $a \sqsubseteq a'$ and $a \neq a'$.

The first and more obvious order theoretical property that termination establishes is that any terminating preorder is a partial order. Moreover, if $(A, \sqsubseteq)$ is terminating then for any (possibly infinite) sequence

$$a_0 \sqsubseteq a_1 \sqsubseteq \cdots \sqsubseteq a_n \sqsubseteq \cdots$$

there exists an $N \geqslant 0$ such that, for all i, $a_i \sqsubseteq a_N$. if $i < N$ then $a_i \sqsubseteq a_{i+1} \sqsubseteq \cdots \sqsubseteq a_N$ and so $a_i \sqsubseteq a_N$; if $i \geqslant N$, then $a_i = a_N$ and so $a_i \sqsubseteq a_N$. In other words, for any chain C there exists a finite sub-sequence C' of C that dominates it. This can be used to show that any terminating preorder is a a complete partial order. And thus we may establish that any preordered Σ-algebra whose carrier is terminating is in fact a complete partially ordered Σ-algebra.

A **covering** of a preorder $(A, \sqsubseteq)$ is a subset $X \subseteq A$ such that for every $a \in A$ there exists a $x \in X$ such that $a \sqsubseteq x$. A covering X is **minimal** if no proper subset of it is a covering of $(A, \sqsubseteq)$ (or equivalently, of $(X, \sqsubseteq)$). The relation of terminating relations with the existence of minimal coverings is expressed by:

Proposition 3.1 *If $(A, \sqsubseteq)$ is terminating then $\mathcal{N}_A = \{a \in A \mid a \text{ is maximal}\}$ is a minimal covering of $(A, \sqsubseteq)$.*

The following examples show that this implication is proper:

Example 3.2 *Consider the preorder*

$$\begin{tikzcd} & & a & & \\ a_0 \arrow[r] \arrow[urr] & a_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & a_n \arrow[r] \arrow[ul] & \cdots \end{tikzcd}$$

There exists a minimal covering of $\{a_i\}_{i\geqslant 0} \cup \{a\}$, namely the set $\{a\}$. However the preorder is non-terminating.

Example 3.3 *Other examples of non-terminating preorders where there exists a minimal covering are:*

$$\begin{tikzcd} & & a & & \\ a_0 \arrow[r] \arrow[urr] \arrow[drr] & a_1 \arrow[r] \arrow[ur] \arrow[dr] & \cdots \arrow[r] & a_n \arrow[r] \arrow[ul] \arrow[dl] & \cdots \\ & & a' & & \end{tikzcd}
\qquad
\begin{tikzcd} & & a & & & \\ a_0 \arrow[r] \arrow[urr] & a_1 \arrow[r] \arrow[dr] & \cdots \arrow[r] & a_{2n} \arrow[r] \arrow[ul] & a_{2n+1} \arrow[r] \arrow[dll] & \cdots \\ & & a' & & & \end{tikzcd}$$

In both cases the set $\{a, a'\}$ is a minimal covering of the depicted preorders and these are non-terminating.

The main difference between these and Example 3.2 is that in the first one the minimal covering has an extra property: for each $a \in A$ there exists a unique element in $\mathcal{N}$ above a. This motivates the following definition:

Definition 3.4 *Given a preorder* $(A, \sqsubseteq)$, *a minimal covering* X *of* A *is a* **uniquely minimal covering** *if for any element* a *of* A *there exists a unique element* x *in* X *such that* $a \sqsubseteq x$.

These definitions of coverings, minimal coverings, and uniquely minimal coverings, correspond to the definitions of floorings in [9]. The only difference is that we are defining these concepts with respect to an arbitrary preorder rather than for the particular case of the underlying preorder of a given category. The importance of uniquely minimal coverings is that:

Lemma 3.5 *Given an preordered* Σ*-algebra* $(A, \sqsubseteq)$ *and a uniquely minimal covering* $\mathcal{N}$ *of* A *then the unique mapping* $\mathrm{nf}_A : A \to \mathcal{N}$ *satisfying* $a \sqsubseteq \mathrm{nf}_A(a)$ *for any element* a *in* A. *also satisfies: (1) if* $a \sqsubseteq a'$ *then* $\mathrm{nf}_A(a) = \mathrm{nf}_A(a')$; *(2) for any element* $a \in A$, $\mathrm{nf}_A(\mathrm{nf}_A(a)) = \mathrm{nf}_A(a)$; *(3) for any element* $a \in \mathcal{N} \subseteq A$, $\mathrm{nf}_A(a) = a$.

Proof. Note that nf_A sends each element a of A to the unique element a' of $\mathcal{N}$ such that $a \sqsubseteq a'$. Then, as $a \sqsubseteq a' \sqsubseteq \mathrm{nf}_A(a')$, and $\mathrm{nf}_A(a)$ is the unique element of $\mathcal{N}$ above a, then $\mathrm{nf}_A(a) = \mathrm{nf}_A(a')$, proving (1). Using the above and the fact that $a \sqsubseteq \mathrm{nf}_A(a)$ we have that $\mathrm{nf}_A(\mathrm{nf}_A(a)) = \mathrm{nf}_A(a)$, proving (2). Finally (3) follows because $a \in \mathcal{N}$ and $a \sqsubseteq a$. □

Using this Lemma we can show that

Proposition 3.6 *Given a preordered* Σ*-algebra* A *and a uniquely minimal covering* $\mathcal{N}$ *of its carrier then we can make* $\mathcal{N}$ *into a* Σ*-algebra: for each* $\sigma \in \Sigma_n$ *we define* $\mathcal{N}_\sigma$ *as*

$$\mathcal{N}_\sigma(\overline{a}_1, \ldots, \overline{a}_n) = \mathrm{nf}_A(A_\sigma(\overline{a}_1, \ldots, \overline{a}_n))$$

for all elements $\overline{a}_i \in \mathcal{N}$. *Moreover the mapping* nf_A *as defined above is a* Σ*-homomorphism from* A *(when seen as a* Σ*-algebra) to* $\mathcal{N}$.

A preorder $(A, \sqsubseteq)$ is **confluent** if, whenever $a \sqsubseteq a_1$ and $a \sqsubseteq a_2$ there exists a' such that $a_1 \sqsubseteq a'$ and $a_2 \sqsubseteq a'$.

Remark 3.7 *If* $(A, \sqsubseteq)$ *is confluent then, for every* $a \in A$, *the set* $\{a' \mid a \sqsubseteq a'\}$ *is directed.*

Hence, if $(A, \sqsubseteq)$ is also complete, the least upper bounds of these sets exist. The following Proposition shows how uniquely minimal coverings are related to confluent preorders

Proposition 3.8 *Given a preorder* $(A, \sqsubseteq)$, *a minimal covering* $\mathcal{N}$ *of* A *is uniquely minimal iff* $\sqsubseteq$ *is confluent.*

Proof. We prove the 'if' part by contradiction. Assume that $\mathcal{N}$ is a minimal covering of A and that there exists $a \in A$ for which there exist $a_1, a_2 \in \mathcal{N}$ such that $a \sqsubseteq a_1$ and $a \sqsubseteq a_2$ and $a_1 \neq a_2$. Then there exists $a' \in A$ such that $a_1 \sqsubseteq a'$ and $a_2 \sqsubseteq a'$. Let a'' be an element of $\mathcal{N}$ such that $a' \sqsubseteq a''$; this element exists because $\mathcal{N}$ is a covering of A. Then the set $(\mathcal{N} - \{a_1, a_2\}) \cup \{a''\}$ is a covering of A and is a proper subset of $\mathcal{N}$ because $a_1 \neq a_2$. Thus $\mathcal{N}$ is not a minimal covering contradicting the assumption. Hence $a_1 = a_2$. For the 'only-if' part assume that for some $a, a_1, a_2 \in A$ we have that $a \sqsubseteq a_1$ and $a \sqsubseteq a_2$. Let $a_1', a_2' \in \mathcal{N}$ be such that $a_1 \sqsubseteq a_1'$ and $a_2 \sqsubseteq a_2'$; these elements exist because $\mathcal{N}$ is a covering of A. But then $a \sqsubseteq a_2'$ and $a \sqsubseteq a_1'$. As there exists a unique element of $\mathcal{N}$ above a, $a_1' = a_2'$ and so $a_1 \sqsubseteq a_1'$ and $a_2 \sqsubseteq= a_1'$. □

Uniquely minimal coverings need not be unique. Consider the preorder

$$\begin{tikzcd} a \arrow[r] & b \arrow[d] \\ d \arrow[ur] & c \arrow[l] \end{tikzcd}$$

Any of the sets $\{b\}$, $\{c\}$, or $\{d\}$ is a uniquely minimal covering of $\{a, b, c, d\}$. However,

Proposition 3.9 *Given a preordered Σ-algebra A and two uniquely minimal coverings $\mathcal{N}$ and $\mathcal{N}'$ of its carrier, the associated Σ-algebras $\mathcal{N}$ and $\mathcal{N}'$ are isomorphic.*

But in the case where A is a partial order, uniquely minimal coverings are indeed unique. We can now relate the completeness of a partial order with the existence of a uniquely minimal covering.

Proposition 3.10 *If A is a complete and confluent partial order then there exists a uniquely minimal covering of A.*

Proof. for any a in A let $\overline{a}$ be defined as $\overline{a} = \bigsqcup\{a' \mid a \sqsubseteq a'\}$. That A is confluent ensures that this set is directed and as A is complete that lub exists. Define $\mathcal{N}$ as $\mathcal{N} = \{\overline{a} \mid a \in A\}$. We show that $\mathcal{N}$ is a uniquely minimal covering of A by showing that (1) it is a covering, (2) it is minimal, and (3) it is uniquely minimal. To prove (1), let a in A. Then $\overline{a}$ in $\mathcal{N}$ and $a \sqsubseteq \overline{a}$. Now, let $\mathcal{N}' \subseteq \mathcal{N}$ be another covering of A. We show that $\mathcal{N} \subseteq \mathcal{N}'$ and thus $\mathcal{N} = \mathcal{N}'$, proving (2). Let $n \in \mathcal{N}$; then, for some $a \in A$, $n = \bigsqcup\{a' \mid a \sqsubseteq a'\}$. Let $n' \in \mathcal{N}'$ be such that $n \sqsubseteq n'$ (n' exists because by assumption $\mathcal{N}'$ is a covering of A and $n \in \mathcal{N} \subseteq A$); then $a \sqsubseteq n \sqsubseteq n'$ and so n' belongs to the set $\{a' \mid a \sqsubseteq a'\}$. As n is the maximum of this set, $n' \sqsubseteq n$. The anti-symmetry of $\sqsubseteq$ establishes that $n = n'$ and thus $n \in \mathcal{N}'$. (3) follows directly from Proposition 3.8 $\square$

Lemma 3.11 *If A is a complete partial order and $\mathcal{N}$ is a uniquely minimal covering of A, then the mapping $\mathrm{nf}_A : A \to \mathcal{N}$ is a continuous mapping from A to the complete partial order $(\mathcal{N}, =)$.*

Proof. Let Δ be a directed subset of A whose lub is d. Note that as $\mathcal{N}$ is a uniquely minimal covering, there exists $n \in \mathcal{N}$ such that $d \sqsubseteq n = \mathrm{nf}_A(d)$ for all $d \in \Delta$. Moreover $d \sqsubseteq n = \mathrm{uf}_A(d)$. Then,

$$\begin{aligned} \bigsqcup\{\mathrm{nf}_A(\delta) \mid \delta \in \Delta\} &= \bigsqcup\{n\} = n = \mathrm{nf}_A(d) \\ &= \mathrm{nf}_A(\bigsqcup \Delta) \end{aligned}$$

$\square$

The results we have presented so far can be extended to some kind of non-termination, namely for globally finite preorders. A binary relation $\sqsubseteq$ over A is **globally finite** if for any $a \in A$ the set $\{a' \mid a \sqsubseteq a'\}$ is finite.

If $\sqsubseteq$ is a globally finite binary relation, then for any sequence

$$a_0 \sqsubseteq a_1 \sqsubseteq \cdots \sqsubseteq a_n \sqsubseteq \cdots$$

there exists an $N \geqslant 0$ such that $a_i \sqsubseteq a_N$ for all $i \geqslant 0$. This allows us to show that any globally finite preorder is in fact a complete preordered set. And so, we can show that a preordered Σ-algebra whose carrier is globally finite is a complete preordered Σ-algebra. The main difference from the terminating case is that here we cannot ensure that $\sqsubseteq$ is a partial order.

If $(A, \sqsubseteq)$ is a globally finite and confluent preorder we define $(\!|\_|\!)$ as the mapping from A to $A/_{\simeq}$ that sends each $a \in A$ to the class

$$(\!|a|\!) = \bigsqcup\{[a']_{\simeq} \mid a \sqsubseteq a'\}$$

The well-definedness of $(\!|\_|\!)$ follows from the fact that if $\sqsubseteq$ is confluent the set $\{a' \mid a \sqsubseteq a'\}$ is directed and so is $\{[a']_{\simeq} \mid a \sqsubseteq a'\}$; if $\sqsubseteq$ is globally finite, $A/_{\simeq}$ is complete and so that lub exists.

Proposition 3.12 *Let A be a preordered Σ-algebra and C be a minimal covering of A. Then the set*

$$C/_{\simeq} = \{[c]_{\simeq} \in A/_{\simeq} \mid c \in C\}$$

is a minimal covering of $A/_{\simeq}$. Moreover, if C is uniquely minimal $C/_{\simeq}$ is the (unique) uniquely minimal covering of $A/_{\simeq}$.

The proof of this Proposition, shows that we could also prove that if A is a preorder such that there exists a (uniquely) minimal covering of $A/_{\simeq}$, then there exists a (uniquely) minimal covering of A. This proof uses the Axiom of Choice. It can also be shown that in fact it is equivalent to this Axiom. Another use of the Axiom of Choice, or more precisely of Zorn's Lemma, is in the proof of the following generalization of Proposition 3.1:

Proposition 3.13 *If $(A, \sqsubseteq)$ is a complete preorder then there exists a minimal covering of it.*

From which we can immediately establish that:

Corollary 3.14 *If $(A, \sqsubseteq)$ is a globally finite preorder then there exists a minimal covering of it.*

We can strengthen this result for the case of confluent preorders:

Proposition 3.15 *If $(A, \sqsubseteq)$ is a globally finite and confluent preorder then there exists a uniquely minimal covering of it, and a unique uniquely minimal covering of the partial order $A/_{\simeq}$.*

These results can again be lifted to algebras.

We have seen in the previous sections how termination ensures the existence of a minimal covering (which is uniquely minimal in the case where $\sqsubseteq$ is confluent). We have also seen that in some cases of non-termination (global finiteness) these properties still hold. But in general they don't: just consider the natural numbers ordered in the usual way; then the chain

$$\langle 0, 1, 2, \ldots, n, \ldots \rangle$$

doesn't have any upper bound; moreover there is no minimal covering of ω.

4 Applications to Term Rewrite Systems

Given a signature Σ, a **Σ-rewrite rule** (or simply rewrite rule) is a triple (X, l, r) where X is a set of variables disjoint from Σ and l and r are $(\Sigma \cup X)$-terms. It is often required that the variables that occur in r also occur in l and that l is not a single variable. In the present work we do not impose either of these restrictions. We often write a rewrite rule in the form $(\forall X)\ l \to r$. A **term rewrite system** (TRS) is just a set rewrite rules.

Given a TRS $\mathcal{R}$ and a ground signature X, the **one-step rewrite** relation is denoted by $\to_{\mathcal{R}}$ and defined as the least relation over $T_{\Sigma}(X)$ such that

- for all rewrite rules $(\forall A)\ l \to r$ in $\mathcal{R}$ and assignments $\theta : A \to T_\Sigma(X)$, $\overline{\theta}(l) \to_{\mathcal{R}} \overline{\theta}(r)$,
- for each operation symbol $\sigma \in \Sigma_n$, $\sigma(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_n) \to_{\mathcal{R}} \sigma(t_1, \ldots, t_{k-1}, t'_k, t_{k+1}, \ldots, t_n$ whenever $t_k \to_{\mathcal{R}} t'_k$.

We define the rewrite relation $\to^*_{\mathcal{R}}$ as the reflexive and transitive closure of $\to_{\mathcal{R}}$, and say that a rewrite system $\mathcal{R}$ is **confluent** if for all X, $\to_{\mathcal{R}}$ is confluent.

The similarities between the definition of a rewrite rule and an inequation are very important. In fact, when we see each rule $(\forall X)\ l \to r$ of a TRS $\mathcal{R}$ as the Σ-inequation $(\forall X)\ l \sqsubseteq r$, the definition of the rewrite relation establishes that $(T_\Sigma(X), \to^*_{\mathcal{R}})$ is a preorder. Moreover, this enables us to define the preordered Σ-algebra $OT_{\Sigma,\mathcal{R}}(X)$ as having carrier $(T_\Sigma(X), \to^*_{\mathcal{R}})$ and, for each operation symbol σ, the corresponding function $(OT_{\Sigma,\mathcal{R}}(X))_\sigma$ defined as

$$(OT_{\Sigma,\mathcal{R}}(X))_\sigma(t_1, \ldots, t_n) = (T_\Sigma(X))_\sigma(t_1, \ldots, t_n) = \sigma(t_1, \ldots, t_n)$$

That these functions are monotonic follows because $\sigma(t_1, \ldots, t_n) \to^*_{\mathcal{R}} \sigma(t'_1, \ldots, t'_n)$, whenever $t_i \to^*_{\mathcal{R}} t'_i$ for all i. The above definition of the rewrite relation shows that $OT_{\Sigma,\mathcal{R}}(X)$ satisfies the inequations in $\mathcal{R}$, allowing us to establish the following completeness result:

Theorem 4.1 *If, for some set of inequations $\mathcal{R}$, an inequation $(\forall X) t_1 \sqsubseteq t_2$ is satisfied by all preordered Σ-algebras that satisfy $\mathcal{R}$, then $t_1 \to^*_{\mathcal{R}} t_2$.*

Moreover,

Proposition 4.2 *Let A be a preordered Σ-algebra satisfying $\mathcal{R}$ and $\theta : X \to A$; then the unique Σ-homomorphism $\overline{\theta} : T_\Sigma(X) \to A$ that extends θ is a monotonic mapping from $OT_{\Sigma,\mathcal{R}}(X)$ to A.*

This allows us to establish:

Theorem 4.3 *Given a ground signature X, a preordered Σ-algebra A satisfying $\mathcal{R}$, and an assignment $\theta : X \to A$, there exists a unique ordered Σ-homomorphism $\overline{\theta} : OT_{\Sigma,\mathcal{R}}(X) \to A$ that extends θ.*

And thus, for any preordered Σ-algebra A satisfying $\mathcal{R}$, there exists a unique monotonic Σ-homomorphism from $OT_{\Sigma,\mathcal{R}}$ to A. In other words, $OT_{\Sigma,\mathcal{R}}$ is **initial** in the class of preordered Σ-algebras that satisfy the inequations of $\mathcal{R}$.

These results point out the benefit of treating rules as inequations. They do not depend on confluence, termination, or any other property of $\mathcal{R}$, thus representing the answer to the problem of providing an algebraic semantics for rewrite rules with a much wider field of application than the traditional (equational) solution (e.g., Huet and Oppen [15], Goguen [8]). A typical example appears in [19]: the specification of a non-deterministic operation CHOICE described by the two rules

$$(\forall x, y) \quad \text{CHOICE}(x, y) \to x$$
$$(\forall x, y) \quad \text{CHOICE}(x, y) \to y$$

It does not make sense to interpret these rules as equations: only the trivial model satisfies them! But taking this new approach, models of this rewrite system are preordered algebras where all the values of the expressions E and E' are possible values of the expression CHOICE(E, E'). The initiality of $OT_{\Sigma,\mathcal{R}}$ states that for any such model there exists a **unique** way of interpreting a term, and that this interpretation is in fact monotonic. Furthermore, Theorem 2.4 can be obtained as a corollary of this last one. Notice that the preorder $(T_\Sigma(X), \to^*_{\mathcal{R}})$ is nothing more then the underlying preorder of the category $\mathcal{T}_{\mathcal{R}}(X)$ defined by Meseguer [19]. Hence, the differences with Meseguer's approach are that we do not distinguish different rewritings between two terms (for a careful comparison of these two formalisms see Section 3.6 of [19]). The simplicity of our definitions is reflected in the proof of the Completeness Theorem above and of the following soundness result:

Theorem 4.4 *Let $\mathcal{R}$ be a TRS, X a ground signature and $t_1, t_2 \in T_\Sigma(X)$. Then, for any preordered Σ-algebra A that satisfies the inequations in $\mathcal{R}$, if $t_1 \to^*_{\mathcal{R}} t_2$ then A satisfies the inequation $(\forall X)\ t_1 \sqsubseteq t_2$.*

Proof. Let $\theta : X \to A$ be any assignment. As $\overline{\theta}$ is monotonic then $\overline{\theta}(t_1) \sqsubseteq_A \overline{\theta}(t_2)$. □

4.1 Terminating Rewrite Systems

A TRS is **terminating** if the one step rewrite relation is a terminating relation. This, in conjunction with the results of the previous section, allows us to conclude that, given a terminating TRS $\mathcal{R}$ and an arbitrary ground signature X disjoint from Σ, the preorder $(T_\Sigma(X), \to^*_{\mathcal{R}})$ is complete, and $OT_{\Sigma,\mathcal{R}}(X)$ is a continuous Σ-algebra satisfying the inequations in $\mathcal{R}$. Furthermore, if A is a continuous preordered Σ-algebra satisfying the inequations of $\mathcal{R}$, and $\theta : X \to A$ is an assignment, then the unique monotonic Σ-homomorphism that extends θ is continnous. This enables us to prove the following freeness result:

Theorem 4.5 *Let $\mathcal{R}$ be a terminating TRS. Then, given a ground signature X, a continuous Σ-algebra A satisfying $\mathcal{R}$, and an assignment $\theta : X \to A$, there exists a unique continuous Σ-homomorphism $\overline{\theta} : OT_{\Sigma,\mathcal{R}}(X) \to A$ that extends θ.*

Thus, if $\mathcal{R}$ is a terminating TRS, $OT_{\Sigma,\mathcal{R}}$ is initial in the class of continuous Σ-algebras that satisfy the inequations of $\mathcal{R}$. We can now use the results of the previous section to show:

Proposition 4.6 *If $\mathcal{R}$ is confluent and $(T_\Sigma(X), \to^*_{\mathcal{R}})$ is a continuous Σ-algebra, then for each term t, the normal form $[\![t]\!]_{\mathcal{R}}$, if it exists, is defined as $[\![t]\!]_{\mathcal{R}} = \bigsqcup\{t' \mid t \to^*_{\mathcal{R}} t'\}$*

Proposition 4.7 *If $\mathcal{R}$ is a terminating and confluent TRS there exists a (unique) uniquely minimal covering of the set of terms.*

This minimal covering is exactly the set of normal forms - $\mathcal{N}_{\Sigma,\mathcal{R}}(X)$. We can now use Proposition 3.8 to justify the following:

Definition 4.8 *If $\mathcal{R}$ is confluent and $(T_\Sigma(X), \to^*_{\mathcal{R}})$ is a continuous Σ-algebra, we define the Σ-algebra $\mathcal{N}_{\Sigma,\mathcal{R}}(X)$ as having as carrier the set $\mathcal{N}_{\Sigma,\mathcal{R}}(X)$ and for eaeh $\sigma \in \Sigma_n$, the corresponding operation in $\mathcal{N}_{\Sigma,\mathcal{R}}(X)$ is defined as*

$$(\mathcal{N}_{\Sigma,\mathcal{R}}(X))_\sigma(\overline{t}_1, \ldots, \overline{t}_n) = [\![\sigma(\overline{t}_1, \ldots, \overline{t}_n)]\!]_{\mathcal{R}}$$

for all $\overline{t}_i \in \mathcal{N}_{\Sigma,\mathcal{R}}(X)$

Using Proposition 3.8 again, we can show that

Lemma 4.9 *The mapping $[\![\_]\!]_{\mathcal{R}} : T_\Sigma(X) \to \mathcal{N}_{\Sigma,\mathcal{R}}(X)$ as defined above is a Σ-homomorphism.*

This result, which was proved using only the order theoretical properties of confluent and terminating systems, allows us to show a basic property of normal forms – that, for any $\sigma \in \Sigma_n$ and terms $t_i \in T_\Sigma(X)$,

$$\begin{aligned}\llbracket\sigma(t_1,\ldots,t_n)\rrbracket_{\mathcal{R}} &= (N_{\Sigma,\mathcal{R}}(X))_\sigma(\llbracket t_1\rrbracket_{\mathcal{R}},\ldots,\llbracket t_n\rrbracket_{\mathcal{R}})\\ &= \llbracket\sigma(\llbracket t_1\rrbracket_{\mathcal{R}},\ldots,\llbracket t_n\rrbracket_{\mathcal{R}})\rrbracket_{\mathcal{R}}\end{aligned}$$

Another property that follows from these order theoretical resnlts is well known [8]:

Theorem 4.10 *If $\mathcal{R}$ is a terminating and confluent TRS, the algebra $\mathcal{N}_{\Sigma,\mathcal{R}}(X)$ is initial in the class of Σ-algebras that satisfy the equations of $\mathcal{R}$.*

The previous section pointed out how rewriting is naturally linked with preordered algebras. We showed here which order theoretical properties are associated with termiuating systems. These include the existence of a minimal covering of the preorder $(T_\Sigma(X), \rightarrow^*_{\mathcal{R}})$ and the fact that this preorder is a complete partial order.

The rewrite relation partitions the set of terms into *connected components*. Each of these compouents corresponds to an equivalence class when we forget the orientation of the rules, that is, when we consider the reflexive-transitive-symmetric closure of $\rightarrow_{\mathcal{R}}$, i.e., wheu we see rules as equations. Termination ensures that we can fiud a minimal covering of the set of terms. This minimal covering corresponds exactly to the set of values mentioned in the Introduction; the fact that it is a covering means that every term has at least one value, i.e., that every term is computable. This minimal covering is composed of the maximal elements of each connected component: for a given term a maximal elemeut of the component where it lies is a value of it. Additionally, if the system is couflueut then this coveriug is uuiquely minimal, meaning that maximal elements in each componeut are unique. Tbis implies that each term has a unique value.

When we see eacb rewrite step as a step iu the computation of a term, and each rewriting sequence as a computation of its first element, the completeness of $OT_{\Sigma,\mathcal{R}}(X)$ means that we will always fiad the result of a computatiou. In tbis perspective, the confluence of a system means that any computation of a particular term t, represented by the set $\Delta^*(t) = \{t' \mid t \rightarrow^*_{\mathcal{R}} t'\}$ will always give the same result – the least upper bound of $\Delta^*(t)$.

The minimal covering referred above is what Bergstra and Tucker [2] call a **traversal** of the quotient induced by the rules when seen as equations. Our starting poiut is the ordered set of terms, rather than that quotient. As a consequence, our minimal covering is uuique whereas their traversals aren't. Still according to these authors, the choice of a particular traversal fixes an operatioual view of the abstract data type defined by the rules (when seen as equations). But this is exactly the point of the present paper – the meauing of rewrite rules is primarily linked with computation. This view is cousistent with the ideas put forward by Meseguer in [19, 18] where it is argued that we sbould see rewrite rules as expressing change in a computational system rather than expressing static properties as equations do. Another related formalism is the concept of **canonical term algebra** [11]. As proved by Goguen [8]. the algebra of normal forms is a canonical term algebra. However this property does not follow from the fact that it is a uniquely minimal covering.

4.2 Globally Finite Systems

A TRS $\mathcal{R}$ is **locally finite** if, for any term t, the set $\{t' \mid t \rightarrow_{\mathcal{R}} t'\}$ is finite, and is **globally finite** if $\rightarrow^*_{\mathcal{R}}$ is globally fiuite. The systems that we are interested in are locally finite: we only consider finite stes of rules and each rule is composed only by finite terms. In these conditions it is straightforward to show that global finitness is a proper generalization of termination. The practical motivation for the study of globally finite rewrite systems was pointed out by Goguen [8] and comes for instance from the difficulty of dealing with a commutative rule: in fact if we add a commutative rule to a terminating systems we end up with a globally finite but not terminating TRS. We show in this section that, with the belp of the kernel operation of preorders we can extend the results presented above to this particular kind of non-termination. As we will see, this process will not be enough to extend these results to arbitrary systems.

From an order theoretical point of view, the main difference between globally finite and terminating systems is that in the former, tbe rewrite relation is no longer a partial order. We can however use the results abont the kernel of a preorder to establish that:

Theorem 4.11 *Let $\mathcal{R}$ be a globally finite TRS. Then, given a ground signature X, a continuous partially ordered Σ-algebra A satisfying $\mathcal{R}$, and an assignment $\theta : X \to A$, there exists a unique continuous Σ-homomorphism $\theta^{\#} : (OT_{\Sigma,\mathcal{R}}(X))/_{\simeq} \to A$ that extends θ, i.e., such that $\theta(x) = \theta^{\#}([x]_{\simeq})$.*

From this Theorem we can immediately establish that $OT_{\Sigma,\mathcal{R}}/_{\simeq}$ is iuitial in the class of continuous Σ-algebras tbat satisfy the ineqnations of $\mathcal{R}$.

We can again use Proposition 3.10 to justify the following:

Definition 4.12 *If $\mathcal{R}$ is a globally finite and confluent TRS, for a given ground signature X disjoint from Σ, we define the Σ-algebra $\mathcal{N}^{\cong}_{\Sigma,\mathcal{R}}(X)$ as having carrier the set*

$$\mathcal{N}^{\cong}_{\Sigma,\mathcal{R}}(X) = \{(\!|t|\!)_{\mathcal{R}} \mid t \in T_{\Sigma}(X)\}$$

and, for each $\sigma \in \Sigma_n$,

$$(\mathcal{N}^{\cong}_{\Sigma,\mathcal{R}}(X))_{\sigma}((\!|t_1|\!)_{\mathcal{R}}, \ldots, (\!|t_n|\!)_{\mathcal{R}}) = (\!|\sigma(t_1, \ldots, t_n)|\!)_{\mathcal{R}}$$

The above observations allow us to prove that

Theorem 4.13 *If $\mathcal{R}$ is a globally finite and confluent TRS, the algebra $\mathcal{N}^{\cong}_{\Sigma,\mathcal{R}}(X)$ is isomorphic to $T_{\Sigma,\mathcal{R}}(X)$. The isomorphism $h : \mathcal{N}^{\cong}_{\Sigma,\mathcal{R}}(X) \to (T_{\Sigma}(X))/_{\equiv_{\mathcal{R}}}$ sends each $(\!|t|\!)_{\mathcal{R}}$ to $[t]_{\equiv}$.*

From this it follows immediately that if $\mathcal{R}$ is a globally finite and confluent TRS, then the algebra $\mathcal{N}^{\cong}_{\Sigma,\mathcal{R}}$ (i.e., $\mathcal{N}^{\cong}_{\Sigma,\mathcal{R}}(\varnothing)$) is initial in the class of Σ-algebras that satisfy the equations of $\mathcal{R}$. This extends Theorem 4.10 for the case of globally finite systems. We could use the results of the previous section to show that for globally finite systems, there exists also a uniquely minimal covering of $OT_{\Sigma,\mathcal{R}}(X)$. This implies that this covering is also isomorphic to $(T_{\Sigma}(X))/_{\equiv_{\mathcal{R}}}$. But in this case we cannot guarantee tbat this minimal covering is a canonical term algebra. We can prove (with a proof along the same lines of the one presented in [11]) that in the case of global finiteness, there exists one such minimal covering.

The ease which this extension was done is due to the fact that we are using very abstract properties of rewriting. As we will see, witb anotber smooth step we can extend to arbitrary confluent systems.

4.3 Non-terminating Rewrite Systems

Other approaches to non-terminating TRS's extend the set of terms with infinite terms, in order that this extended set satisfies these properties. In this paper we use a different method: instead of extending the set of terms we use a different set, the set $\mathcal{R}_\omega(X)$ of term rewriting sequences, to which there exists an injection[2] from the set of terms ($T_\Sigma(X)$): the mapping that sends each term t to the rewriting sequence $\langle t, t, \ldots\rangle$. As we will see, this set fulfills all the desired properties, i.e., the properties of the set of terms in the terminating case.

The major constraint that we will assume is the finiteness of the signatures involved (Σ and X). This has as major consequence the denumerability of the set of terms $T_\Sigma(X)$.

The first observation that we can make about $\mathcal{R}_\omega(X)$ is that it is a ω-complete preordered algebra satisfying the inequations of $\mathcal{R}$. Furthermore, for every ω-continuous Σ-algebra A in these conditions and mapping $\theta : X \to A$, there exists a unique ω-continuous Σ-homomorphism $\theta^\#_\omega : \mathcal{R}_\omega(X)/_\simeq \to A$ that extends θ, i.e., that, for each $x \in X$, $\theta(x) = \theta^\#_\omega([\langle x, x, \ldots\rangle]_\simeq)$. This implies that $\mathcal{R}_\omega/_\simeq$ is initial in the class of ω-continuous Σ-algebras that satisfy the inequations of $\mathcal{R}$. Moreover, if both Σ and X are finite then the preorder $(\mathcal{R}_\omega(X), \ll)$ is complete and the operations of $\mathcal{R}_\omega(X)$ are continuous, allowing us to prove:

Theorem 4.14 *If both Σ and X are finite then for every continuous partially ordered Σ-algebra A satisfying the inequations in $\mathcal{R}$ and mapping $\theta : X \to A$, there exists a unique continuous Σ-homomorphism $\theta^\#_\omega : \mathcal{R}_\omega(X)/_\simeq \to A$ that extends θ, i.e., satisfying $\theta(x) = \theta^*_\omega([\langle x, x, \ldots\rangle]_\simeq)$ for all x in X.*

Again this implies that if Σ is finite, then $\mathcal{R}_\omega/_\simeq$ is initial in the class of continuous Σ-algebras that satisfy the inequations of $\mathcal{R}$. Another implication of the completeness of $\mathcal{R}_\omega$ is that we can use Proposition 3.13 to establish:

Proposition 4.15 *The set*

$$\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X) = \{t \in \mathcal{R}_\omega(X) \mid t \text{ is maximal wrt } \ll\}$$

is a minimal covering of $(\mathcal{R}_\omega(X))/_\simeq$.

Each rewriting sequence can be seen as a computation of its first element. The ordering $\ll$ between these sequences is then a measure of relative accuracy between these computations. For $[s]_\simeq$ in $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X)$ we call each $t \in [s]_\simeq$ a **normalizing rewrite sequence**. Each normalizing sequence $\langle t_0, t_1, \ldots\rangle$ being a maximal element wrt $\ll$, represents a very particular computation: none is more accurate than it. It is therefore a good substitute for the concept of normal form. Note that, unlike the other approaches to this problem, we impose no requirements to the rewrite system in order that these sequences exist.

If $\mathcal{R}$ is a confluent TRS then, for any term rewriting sequence t, the set $\Delta^*_\omega(t) = \{[t']_\simeq \mid t \ll t'\}$ is directed. A direct consequence of this is that the preorder $\mathcal{R}_\omega(X)$ is confluent. Hence, the limit $\bigsqcup \Delta^*_\omega(t)$ exists. This allows us to define, for any confluent TRS $\mathcal{R}$, the mapping $\llparenthesis\_\rrparenthesis_\mathcal{R} : \mathcal{R}_\omega(X)/_\simeq \to \mathcal{R}_\omega(X)/_\simeq$ that sends each class $[t]_\simeq$ to $\bigsqcup \Delta^*_\omega(t)$.

Given a confluent TRS $\mathcal{R}$ we define the S-sorted set $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X)$ as

$$\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X) \;=\; \{\llparenthesis t\rrparenthesis_\mathcal{R} \mid t \in \mathcal{R}_\omega(X)\}$$

[2] Recall that the existence of an injection $\imath : A \to B$ is an abstraction of the fact that A is contained in B.

This set has an important property: it is a uniquely minimal covering of $(\mathcal{R}_\omega(X))/_{\simeq}$. This allows us to use the results of the previous section and define the algebra $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X)$ has having $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X)$ as its carrier, and for each $\sigma \in \Sigma_n$,

$$(\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X))_\sigma(\llbracket t^1\rrbracket_\mathcal{R}, \ldots, \llbracket t^n\rrbracket_\mathcal{R}) = \llbracket(\mathcal{R}_\omega)_\sigma(t^1, \ldots, t^n)\rrbracket_\mathcal{R}$$

Moreover $\llbracket\_\rrbracket_\mathcal{R}$ is a Σ-homomorphism from $\mathcal{R}_\omega(X)$ (when seen as a Σ-algebra) to $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X)$. If we compose this Σ-homomorphism to the one that sends each term t to the class $[t]_\simeq$ we get a Σ-homomorphism from $T_\Sigma(X)$ to $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X)$ that sends each term t to $\llbracket\langle t, t \ldots\rangle\rrbracket_\mathcal{R}$.

Lemma 4.16 *Let $\mathcal{R}$ be a confluent TRS, t a term, and t^1 and t^2 term rewriting sequences such that $t^1_0 \to^*_\mathcal{R} t \;{}_\mathcal{R}^*\!\!\leftarrow t^2_0$. then, if $[a^1] \simeq$ and $[a^2] \simeq$ are arbitrary elements of $\Delta^*_\omega(t^1)$ and $\Delta^*_\omega(t^2)$ respectively, there exists a class $[b]_\simeq \in \Delta^*_\omega(t^1) \cup \Delta^*_\omega(t^2)$ such that both $a^1 \ll b$ and $a^2 \ll b$.*

This shows that in these conditions $\llbracket t^1\rrbracket_\mathcal{R} = \llbracket t^2\rrbracket_\mathcal{R}$. Hence

Theorem 4.17 *If $\mathcal{R}$ is a confluent TRS, the algebra $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(X)$ is isomorphic to $(T_\Sigma(X))/_{\equiv_\mathcal{R}}$.*

From which it follows immediately that $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}$ (i.e., $\mathcal{N}^\omega_{\Sigma,\mathcal{R}}(\varnothing)$) is initial in the class of Σ-algebras that satisfy the equations of $\mathcal{R}$.

Each normalizing sequence $\langle t_0, t_1, \ldots\rangle \in \llbracket t, t, \ldots\rrbracket_\mathcal{R}$, being the least upper bound of the set of computations of t represents a very particular computation: it is at least as accurate as any other! Note that, unlike the other approaches to this problem, confluence is the only requirement that we impose to the rewrite system to guarantee the uniqueness of this set of sequences.

We end our exposition by presenting an example that has an intriguing solution in the other approaches to this problem. Let Σ be defined as $\Sigma_0 = \{0, 1\}$, $\Sigma_1 = \{q\}$, and $\Sigma_n = \varnothing$ for $n > 2$. Consider the TRS's:

$$\mathcal{R}_1 = \{(\forall\varnothing)1 \to q(1), (\forall\varnothing)0 \to q(0)\}$$
$$\mathcal{R}_2 = \{(\forall\varnothing)q(1) \to 1, (\forall\varnothing)q(0) \to 0\}$$

In the equational interpretation of rewriting, these two systems are indistinguishable - the orientation of the rules is irrelevant when we see them as equations. This means that all models of one system are models of the other. The initial model has a two point set as carrier and interprets q as the identity mapping.

In the other approaches to the semantics of non-terminating rewriting these systems have very different interpretations: only the trivial model satisfies $\mathcal{R}_1$ (cf. [7]) whereas the initial model that satisfies $\mathcal{R}_2$ has a two point set as carrier and interprets q as the identity mapping.

Our approach allows us to view these systems in two different perspectives:

- when we see the rules as inequations, the models of $\mathcal{R}_1$ are models of $\mathcal{R}_2$ with the reverse ordering.
- applying the construction described in this section we have that $\mathcal{N}^\omega_{\Sigma,\mathcal{R}_1}(X)$ and $\mathcal{N}^\omega_{\Sigma,\mathcal{R}_2}(X)$ are isomorphic: $\mathcal{N}^\omega_{\Sigma,\mathcal{R}_1}$ has carrier the set $\{[\langle 0, q(0), \ldots, q^n(0), \ldots\rangle]_\simeq, [\langle 1, q(1), \ldots, q^n(1), \ldots\rangle]_\simeq\}$ and $\mathcal{N}^\omega_{\Sigma,\mathcal{R}_2}$ the set $\{0, 1\}$. In both cases q is interpreted as the identity.

5 Conclusions

We have presented an algebraic semantics for rewrite systems that does not depend on any special assumptions about these systems. Our approach views rewrite rules not as equations, but as inequations. This allows us to use (non-confluent) rewrite systems to specify non-determinism. For confluent systems, we have shown how rewriting, even in the non-terminating case, can be seen as an alternative to equational reasoning.

Other approaches to this problem have extended to the non-terminating case by using infinite terms as normal forms of non-terminating computations. The difficulties of these approaches are shown by the counter examples of Kennaway *et al.* [16]. These difficulties have made it hard to apply these formalisms to applications like reactive systems. We feel that the solution presented in this paper paves the way for such applications, along the lines of Hennessy [13] and Meseguer [19, 18]. It was for this reason that we avoided the usual restrictions to the form of the rewrite rules.

One particular difference between our approach and the others referred is the rôle played by the "converging" sequences. Consider the signature Σ with $\Sigma_0 = \{a, b, c\}$, $\Sigma_1 = \{f\}$, and $\Sigma_n = \varnothing$ for all $n > 1$, and the TRS

$$\mathcal{R} = \{(\forall\varnothing)\ a \to b, (\forall\varnothing)\ b \to a, (\forall\varnothing)\ c \to f(c)\}$$

Then both of the sequences $\langle a, b, a, b, \ldots\rangle$ and $\langle c, f(c), f(f(c)), \ldots, f^n(c), \ldots\rangle$ are normalizing sequences, and so are equally important for us. But in the previous approaches, only the second sequence has an important property – it converges. This ensures that c can be assigned a *normal form*, whereas a does not have one. As a result most of the important results of these approaches cannot be applied to systems like the one above. One might think that this restriction is reasonable and desirable; but it rules out some interesting examples of non-terminating processes: just think of a scheduler in an operating system – its behaviour does not converge to any particular state (apart from deadlock in some cases); nevertheless the scheduler is a "respectable" and important part of the operating system, it would be good to study. Finally notice that establishing a convergence criteria is not incompatible with our approach; the difference (or more accurately, the novelty) is that we do not need such a restriction.

One extension of the results presented here is rewriting modulo a set of equations. One approach is to consider the rewrite relation modulo the equations, i.e., to use the quotient induced by those equations as the set of the preorder. This solution follows the lines of Goguen [10] and Meseguer [19]. Another solution is to add for each equation $(\forall X)\ t_1 = t_2$, the rules $(\forall X)\ t_1 \to t_2$ and $(\forall X)\ t_2 \to t_1$. Note that if the system is confluent modulo that set of equations then this new rewrite rewrite system is confluent, so these two approaches give the same results. The ease with which we can treat this extension is a direct consequence of the abstract approach that we have taken.

References

[1] José Barros. *Semantics of Non-terminating Systems through Term Rewriting.* PhD thesis, Oxford University, To appear 1995.

[2] Jan Bergstra and John Tucker. Characterization of computable data types by means of a finite equational specification method. In *Automata, Languages and Programming, Seventh Colloquium*, pages 76–90. Springer-Verlag, July 1980. Lecture Notes in Computer Science,

Volume 85; also, Preprint IW 124, Mathematisch Centrum, Department of Computer Science, Amsterdam, November, 1979.

[3] Steven Bloom. Varieties of ordered algebras. *Journal of Computer and System Sciences*, 13:200–212, 1976.

[4] G. Boudol. Computational semantics of term rewriting systems. In *Algebraic Methods in Semantics*, pages 167–236. Cambridge University Press, 1985.

[5] Y. Chen and M. O'Donnell. Infinite terms and infinite rewritings. In *2nd International Workshop on Conditional and Typed Rewriting Systems*, pages 45–52, 1990.

[6] Bruno Courcelle. Infinite trees in normal form and recursive equations having a unique solution. *Mathematical Systems Theory*, 13:131–180, 1979.

[7] Nachum Dershowitz, Stephane Kaplan, and David Plaisted. Rewrite,rewrite, rewrite, ... *Theoretical Computer Science*, 83(1):71–96, 1991. Preliminary versions appear in *Proceedings of the 16th ACM Symposium on Principles of Programming Languages*, 1989 and in *Proceedings of the 16th EATCS International Colloquium on Automata, Languages and Programming*, Volume 372 of the Lecture Notes in Computer Science, 1989.

[8] Joseph Goguen. How to prove algebraic inductive hypotheses without induction: with applications to the correctness of data type representations. In *Proceedings, Fifth Conference on Automated Deduction*, pages 356–373. Springer-Verlag, 1980. Lecture Notes in Computer Science, Volume 87.

[9] Joseph Goguen. What is unification? – a categorical view of substitution, equation and solution. In *Resolution of Equations in Algebraic Systems*, volume 1. Academic Press, 1989. Preliminary version in *Proceedings, Colloquium on the Resolution of Equations in Algebraic Structures*, held in Lakeway, Texas, USA, May 1987.

[10] Joseph Goguen. *Theorem Proving and Algebra*. MIT Press, To appear 1994.

[11] Joseph Goguen, James Thatcher, and Eric Wagner. An initial algebra approach to the specification, correctness and implementation of abstract data types. Technical Report RC 6487, IBM Watson Research Center, October 1976. Appears in *Current Trends in Programming Methodology, IV*, Raymond Yeh, Ed., Prentice-Hall, 1978, pages 80–149.

[12] Joseph Goguen, James Thatcher, Eric Wagner, and Jesse Wright. Initial algebra semantics and continuous algebras. *Journal of the Association for Computing Machinery*, 24(1):68–95, January 1977.

[13] Matthew Hennessy. *Algebraic Theory of Processes*. MIT Press, 1988.

[14] C. A. R. Hoare. *Communicating Sequential Processes*. Prentice-Hall, 1985.

[15] Gerard Huet and Derek Oppen. Equations and rewrite rules: A survey. In Ronald Book, editor, *Formal Language Theory: Perspectives and Open Problems*, pages 349–405. Academic Press, 1980. Also Technical Report No. STAN-CS-80-785, Computer Science Department, Stanford University.

[16] J. Kennaway, J. W. Klop, M. Sleep, and F. de Vries. Transfinite reductions in orthogonal term rewriting systems. *Lecture Notes in Computer Science*, 488:1–12, 1991. Extended abstract.

[17] Michael Levy and T. S. E. Maibaum. Continuous data types. *SIAM Journal on Computing*, 11(2):201–216, 1982.

[18] N. Martí-Oliet and José Meseguer. Rewriting logic as a logical and semantical framework. Technical report, SRI International, 1994.

[19] José Meseguer. Conditional rewriting logic as a unified model of concurrency. *Theoretical Computer Science*, 96:73–155, 1992. Preliminary version appears as SRI International technical report n. SRI-CSL-90-02R.

[20] T. Naoi and Y. Inagaki. Free continuous algebras and semantics of term rewriting systems. Technical report, Department of Information Science, Nagoya University, 1986.

[21] Maurice Nivat. On the interpretation of polyadic recursive schemes. In *Symposia Mathematica*, volume 15. Academic Press, 1975.

[22] J.-C Raoult and Jean Vuillemin. Operational and semantic equivalence between recursive programs. *Journal of the Association for Computing Machinery*, 27(4):772–796, 1980.

[23] David Turner. Miranda: A non-strict functional language with polymorphic types. In Jean-Pierre Jouannaud, editor, *Functional Programming Languages and Computer Architectures*, pages 1–16. Springer-Verlag, 1985. Lecture Notes in Computer Science, Volume 201.