Random Oracles and Obfuscation

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4 November 16

Random Oracles

- Random oracles (ROs) model ideal hash functions [BR93].
  In the RO model:

  All parties have oracle access to a
  uniformly chosen **random function.**

- ROs enable the security proofs of a wide range of practical and strongly secure cryptosystems: encryption & signature schemes, key exchange, disk encryption, ...

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- ROs enable the security proofs of a wide range of practical and strongly secure cryptosystems: encryption & signature schemes, key exchange, disk encryption, ...
- Standard-model counterparts are often less secure and/or less efficient.

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Scheme $\mathsf{Enc}^{O}(K, M)$:

1. Interpret M as the **description** of a hash function Hash.
2. If $O(x) = \mathsf{Hash}(x)$ for $x = 1 \ldots n$ append K to ciphertexts.
3. Else return a normal/good encryption of M.

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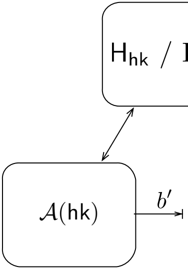
- Lack of a **definition** formalizing "RO-like" behavior.

(Very) Naïve Attempt at Modeling ROs

Call a hash function "IND-RO" if:



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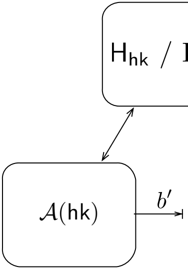

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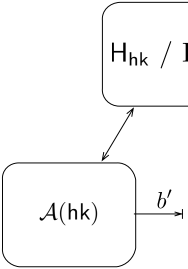

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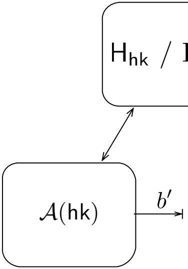

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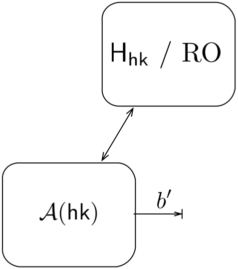

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But observe:

The adversary knows a full input, namely $(\mathsf{hk}, 0)$.

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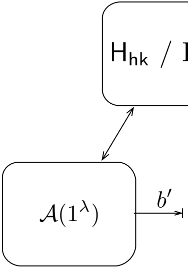

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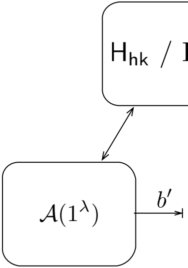

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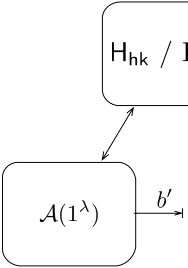

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First idea:

Split $\mathcal{A}$: one part gets hk and the other gets oracle access.

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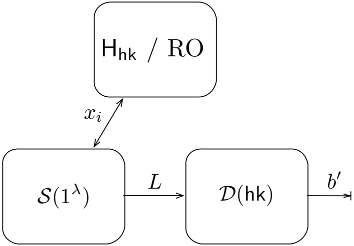

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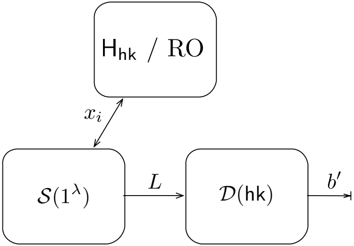

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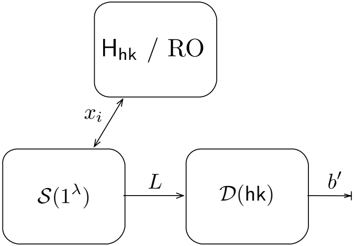

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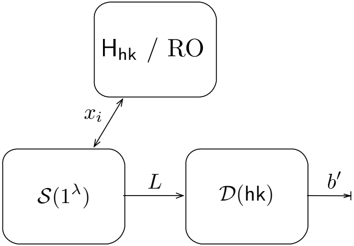

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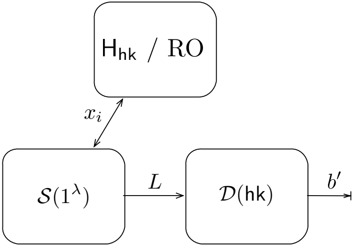

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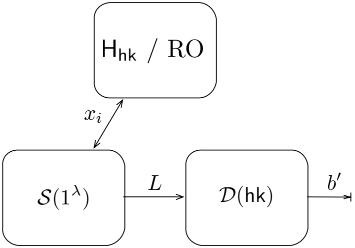
Second idea:

Restrict L: it must not leak any of $\mathcal{S}$'s queries.

Universal Computational Extractors (UCEs) [BHK13a]



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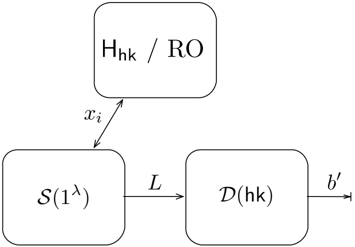

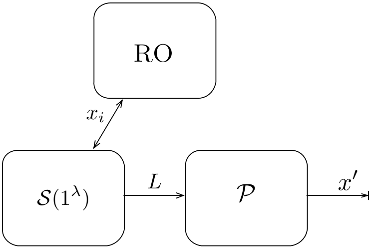

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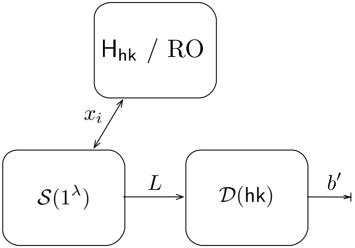

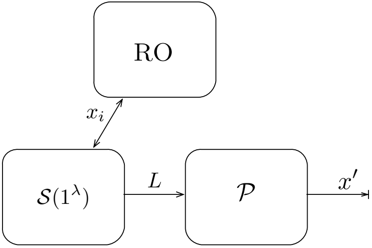

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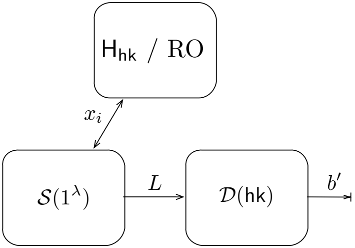

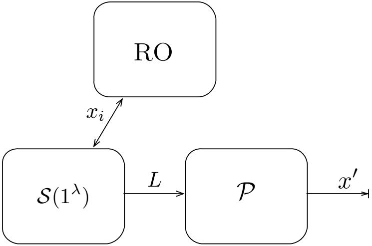

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H is **UCE1 secure** iff: $\mathbf{Adv}^{\mathrm{uce}}_{\mathsf{H},\mathcal{S},\mathcal{D}}(\cdot) \in \textsc{Negl}$ for any **unpredictable** $\mathcal{S}$.

Applications of UCE [BHK13a]

UCE-secure hash functions can instantiate the RO in

- Deterministic PKEs
- RKA and KDM security
- Point-function obfuscation
- Message-locked encryption
- Proofs of storage
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UCEs model many RO-like properties.

Is UCE Security Instantiable?

Suppose we could **strongly obfuscate** H:

- $\mathcal{S}$: Choose a random x. Query x to get y. Leak as L the values

$$y, \quad \mathrm{Obf}(\mathrm{H}(\cdot, x)) \ .$$

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Perhaps a different circuit and/or obfuscator might help?

Indistinguishability Obfuscation (iO)

Let C_0 and C_1 be two **functionally equivalent** circuits:

$$\forall x : C_0(x) = C_1(x) \ .$$

iO **security**: cannot efficiently distinguish obfuscations of such circuits:

$$\mathrm{iO}(C_0) \ \stackrel{c}{\approx} \ \mathrm{iO}(C_1) \ .$$

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Can we use iO to attack UCEs?

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Is $\mathcal{S}$ unpredictable?

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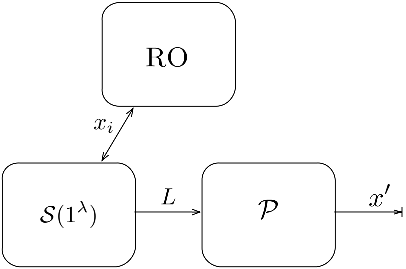

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$$\mathrm{iO}(\mathrm{H}(\cdot, x) \stackrel{?}{=} y) \quad \text{leaks no more than} \quad \mathrm{iO}(\mathrm{Zero})\ ,$$

and the latter is independent of x.